Algorithms and Data Structures
CS-CO-412

David Vernon
Professor of Informatics
University of Skövde
Sweden

david@vernon.eu
www.vernon.eu

Trees

Lecture 7

Topic Overview

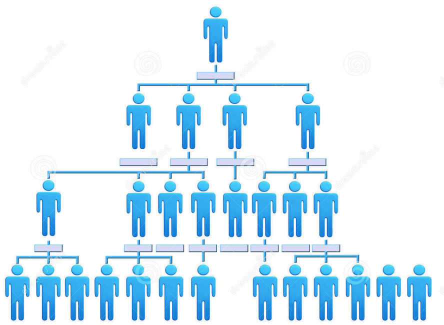
- **Types of trees**
- **Binary Tree ADT**
- Binary Search Tree
- Optimal Code Trees
- Huffman's Algorithm
- Height Balanced Trees
  - AVL Trees
  - Red-Black Trees

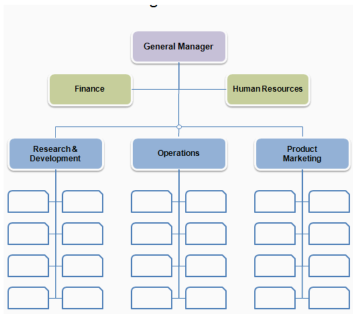
Trees

- Trees are everywhere
- Hierarchical method of structuring data
- Uses of trees:
  - genealogical tree
  - organizational tree
  - expression tree
  - binary search tree
  - decision tree

Uses of Trees

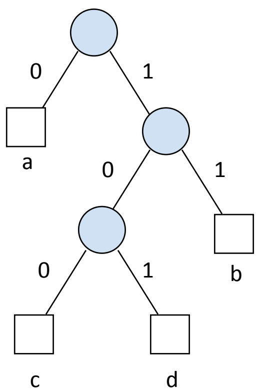
Organization Tree





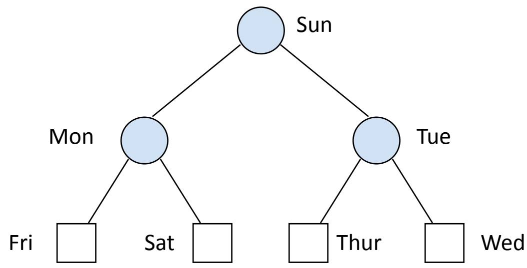
Uses of Trees

Code Tree



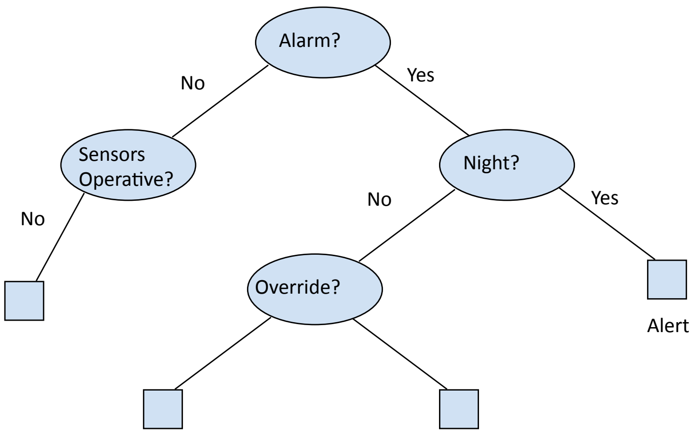
Uses of Trees

Binary Seach Tree



Uses of Trees

Decision Tree



Trees

- Fundamentals
- Traversals
- Display
- Representation
- Abstract Data Type (ADT) approach
- Emphasis on binary tree
- Also multi-way trees, forests, orchards

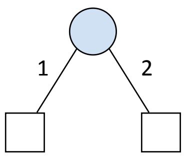
Tree Definitions

- A binary tree T of n nodes, $n \geq 0$,
  - either is empty, if $n = 0$
  - or consists of a root node u and two binary trees $u(1)$ and $u(2)$ of n_1 and n_2 nodes, respectively, such that $n = n + n_1 + n_2$
- We say that $u(1)$ is the first or left subtree of T, and $u(2)$ is the second or right subtree of T

Binary Tree

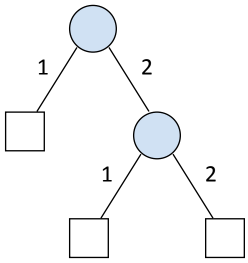
Binary Tree of zero nodes

Binary Tree



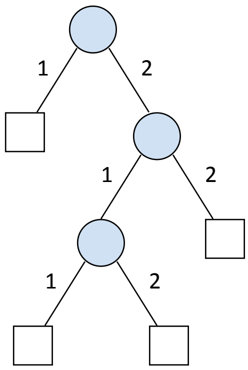
Binary Tree of 1 nodes

Binary Tree



Binary Tree of 2 nodes

Binary Tree



Binary Tree of 3 nodes

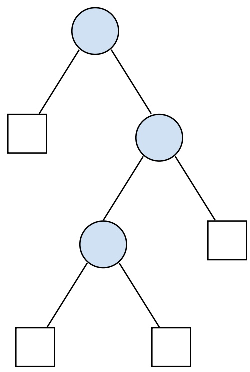
Binary Tree

External nodes - have no subtrees (referred to as leaf nodes)

Internal nodes - always have two subtrees

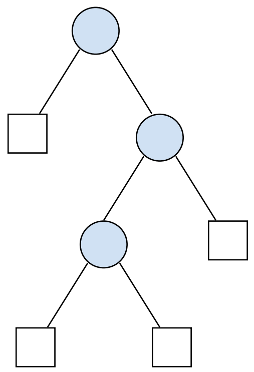
Binary Tree Terminology

- Let *T* be a binary tree with root *u*
- Let *v* be any node in *T*
- If *v* is the root of either *u*(1) or *u*(2), then we say *u* is the parent of *v* and that *v* is the child of *u*
- If *w* is also a child of *u*, and *w* is distinct from *v*, we say that *v* and *w* are siblings



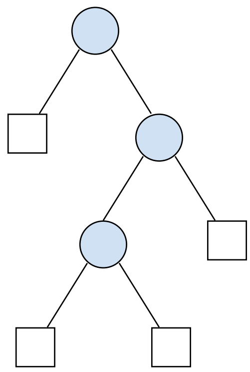
Binary Tree Terminology

- If *v* is the root of *u*(*i*)
- then *v* is the *i*th child of *u*;
  u(1) is the left child and *u*(2) is the right child
- Also have grandparents and grandchildren



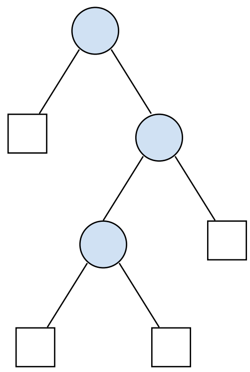
Binary Tree Terminology

- Given a binary tree T of n nodes, $n \geq 0$
- then *v* is a descendent of *u* if either
  - *v* is equal to *u*
    or
  - *v* is a child of some node *w* and *w* is a descendant of *u*
- We write $v\ desc_T\ u$
- v is a proper descendent of *u* if *v* is a descendant of *u* and $v \neq u$



Binary Tree Terminology

- Given a binary tree T of n nodes, $n \geq 0$
- then *v* is a left descendent of *u* if either
  - *v* is equal to *u*
    or
  - *v* is a left child of some node *w* and *w* is a left descendant of *u*
- We write $v\ ldesc_T\ u$
- Similarly we have $v\ rdesc_T\ u$

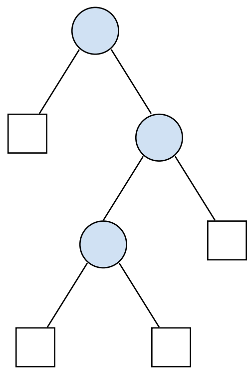


Binary Tree Terminology

- *left*$_T$ relates nodes across a binary tree rather than up and down a binary tree

- Given two nodes u and v in a binary tree T, we say that v is to the left of u if there is a new node w in T such that v is a left descendant of w and u is a right descendant of w

- We denote this relation by *left*$_T$ and write v *left*$_T$ u

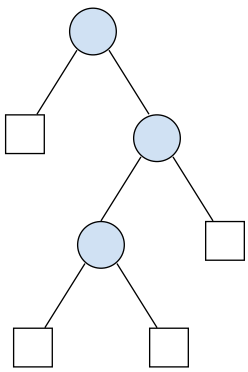


Binary Tree Terminology

- The external nodes of a tree define its frontier
- We can count the number of nodes in a binary tree in three ways:
  - Number of internal nodes
  - Number of external nodes
  - Number of internal and external nodes
- The number of internal nodes is the size of the tree

Binary Tree Terminology

- Let T be a binary tree of size n , $n \geq 0$,
- Then, the number of external nodes of T is $n + 1$



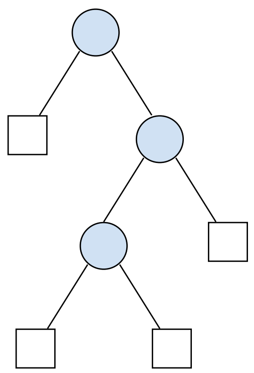
Binary Tree Terminology

- The height of T is defined recursively as

  0 if T is empty and

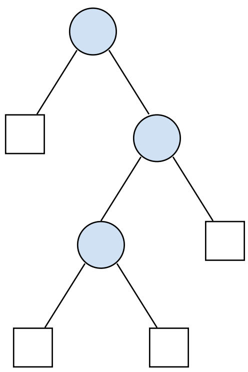
  $1 + max(height(T_1), height(T_2))$ otherwise,
  where T_1 and T_2 are the subtrees of the root.

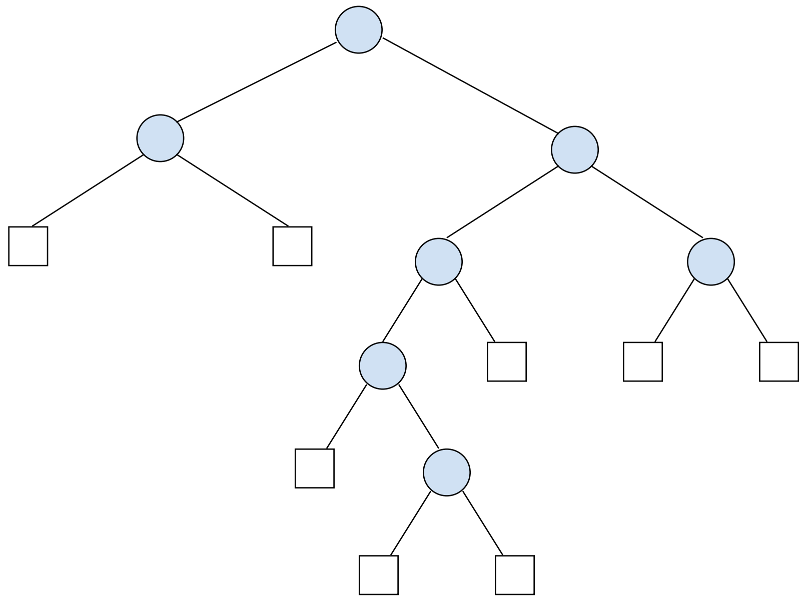
- The height of a tree is the length of a longest chain of descendents



Binary Tree Terminology

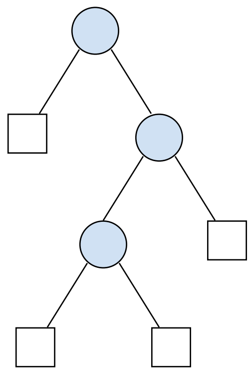
- Height Numbering
  - Number all external nodes 0
  - Number each internal node to be one more than the maximum of the numbers of its children
  - Then the number of the root is the height of *T*
- The height of a node *u* in *T* is the height of the subtree rooted at *u*

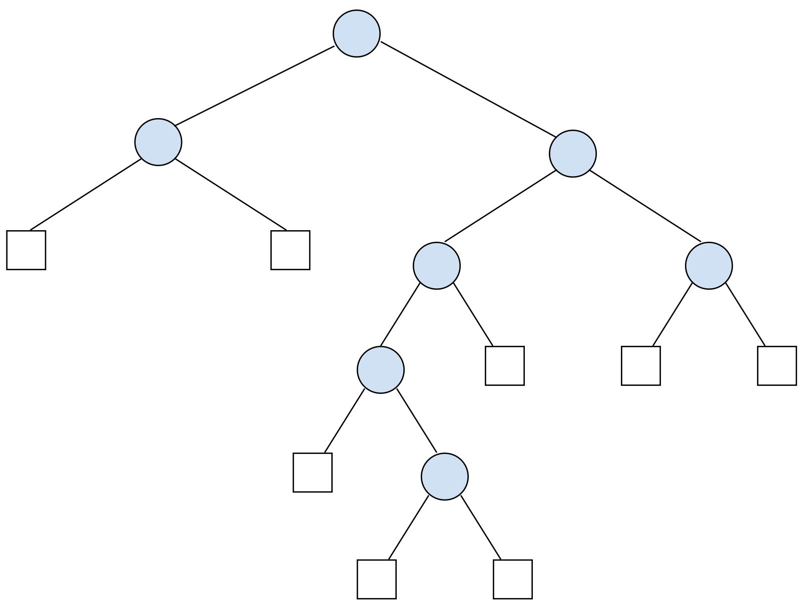




Binary Tree Terminology

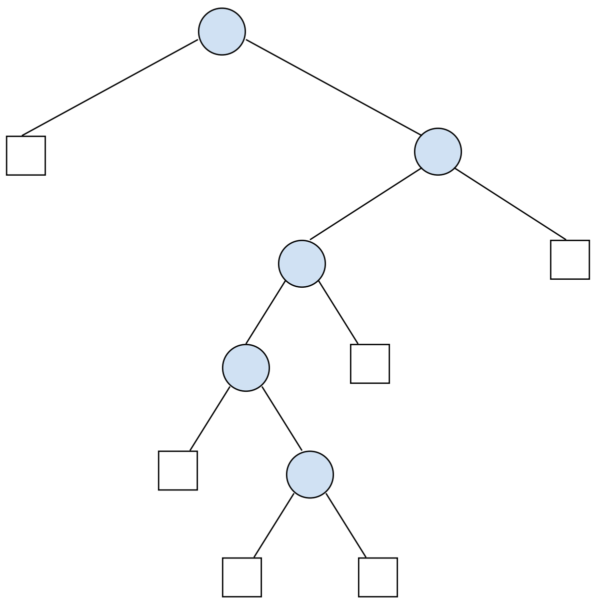
- Levels of nodes
  - The level of a node in a binary tree is computed as follows
  - Number the root node 0
  - Number every other node to be 1 more than its parent
  - Then the number of a node *v* is that node's level
  - The level of *v* is the number of branches on the path from to root to *v*





Binary Tree Terminology

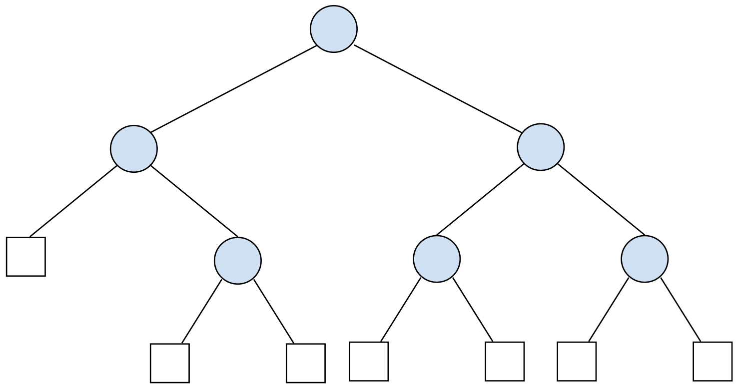
- Skinny Trees
  - every internal node has at most one internal child



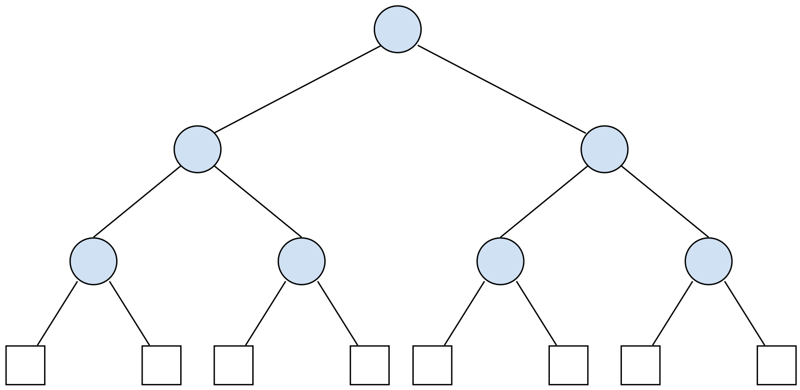
Binary Tree Terminology

- Complete Binary Trees (Fat Trees)
  - the external nodes appear on at most two adjacent levels
  - Perfect Trees: complete trees having all their external nodes on one level
  - Left-complete Trees: the internal nodes on the lowest level is in the leftmost possible position
  - Skinny trees are the highest possible trees
  - Complete trees are the lowest possible trees

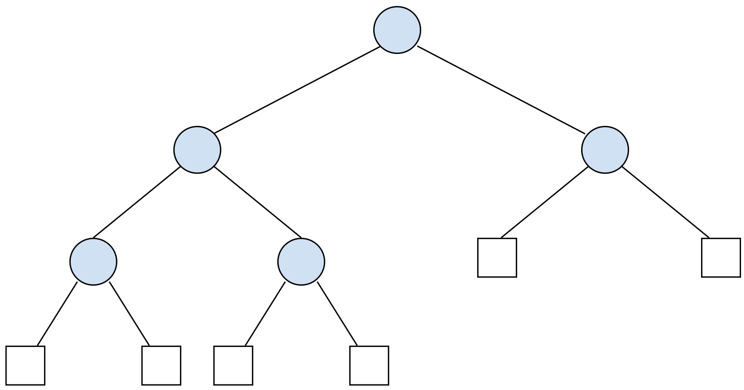
Complete Tree



Perfect Tree



Left-Complete Tree



Binary Tree Terminology

- A binary tree of height $h \geq 0$
  has size at least h
- A binary tree of height at most $h \geq 0$
  has size at most $2^h - 1$
- A binary tree of size $n \geq 0$
  has height at most n
- A binary tree of size $n \geq 0$
  has height at least $\lceil \log (n + 1) \rceil$

Multiway Trees

- Multiway trees are defined in a similar way to binary trees, except that the degree
  (the maximum number of children)
  is no longer restricted to the value 2

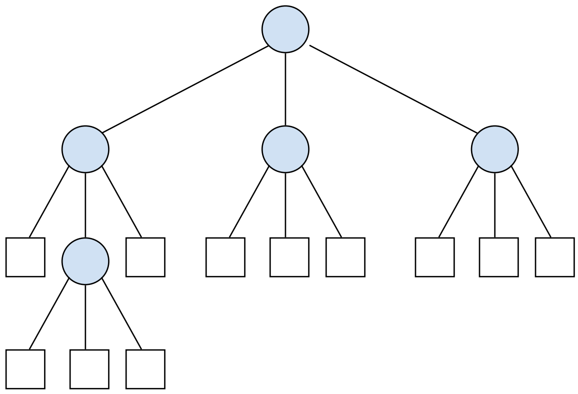
Multiway Trees

- A multiway tree T of n internal nodes, $n \geq 0$,
  - either is empty, if $n = 0$,
  - or consists of
    - a root node u,
    - an integer $d_u \geq 1$, the degree of u,
    - and multiway trees $u(1)$ of n_1 nodes, ..., $u(d_u)$ of n_{d_u} nodes such that $n = 1 + n_1 + ... + n_{d_u}$

Multiway Trees

- A multiway tree T is a d-ary tree,
  for some $d > 0$,
  if $d_u = d$, for all internal nodes u in T

d-ary Tree



Multiway Trees

- A multiway tree T is a (a, b)-tree, if $1 \leq a \leq d_u \leq b$, for all u in T
- Every binary tree is a (2, 2)-tree, and vice versa

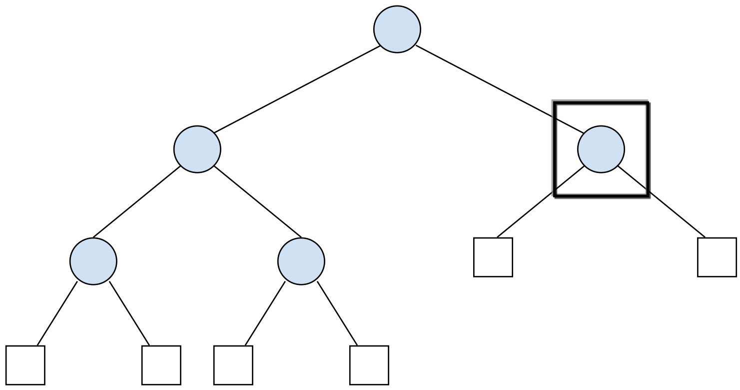
BINARY_TREE & TREE Specification

- So far, no values associated with the nodes of a tree
- Now want to introduce an ADT called BINARY_TREE, which
  - has value of type *intelementtype* associated with the internal nodes
  - has value of type *extelementtype* associated with the external nodes
- These value don't have any effect on BINARY_TREE operations

BINARY_TREE & TREE Specification

- BINARY_TREE has explicit windows and window-manipulation operations
- A window allows us to 'see' the value in a node (and to gain access to it)
- Windows can be positioned over any internal or external node
- Windows can be moved from parent to child
- Windows can be moved from child to parent

Window



BINARY_TREE & TREE Specification

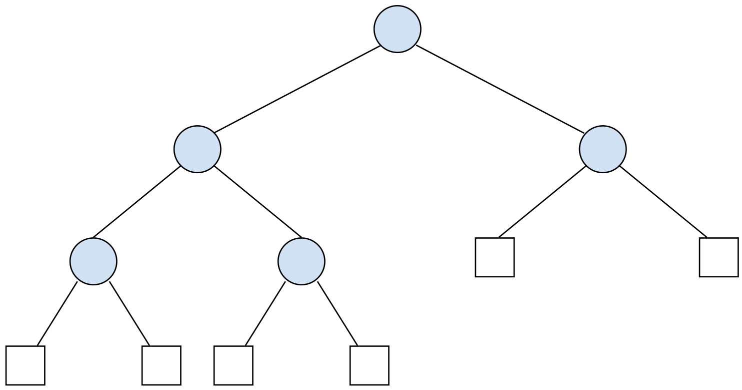
- Let BT denote denote the set of values of BINARY_TREE of *elementtype*
- Let E denote the set of values of type *elementtype*
- Let W denote the set of values of type *windowtype*
- Let B denote the set of Boolean values *true* and *false*

BINARY_TREE Operations

- *Empty*: BT → BT :
  The function *Empty(T)* is an empty binary tree; if necessary, the tree is deleted

- *IsEmpty*: BT → B :
  The function value *IsEmpty*(*T*) is *true* if *T* is empty; otherwise it is false

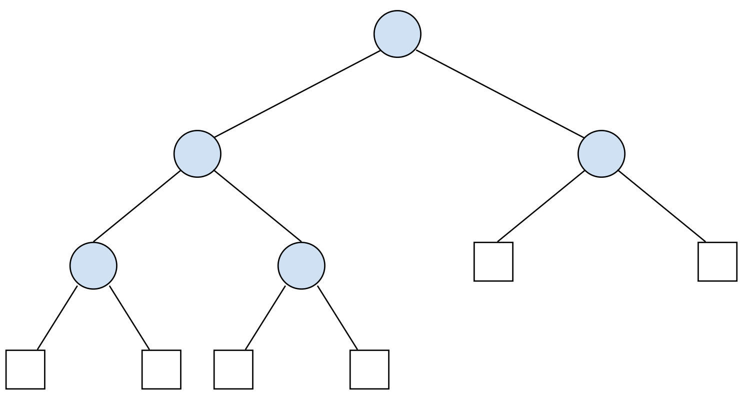
Example



BINARY_TREE Operations

- *Root*: BT $\rightarrow$ W :
  The function value *Root*(*T*) is the window position of the single external node if *T* is empty; otherwise it is the window position of the root of *T*

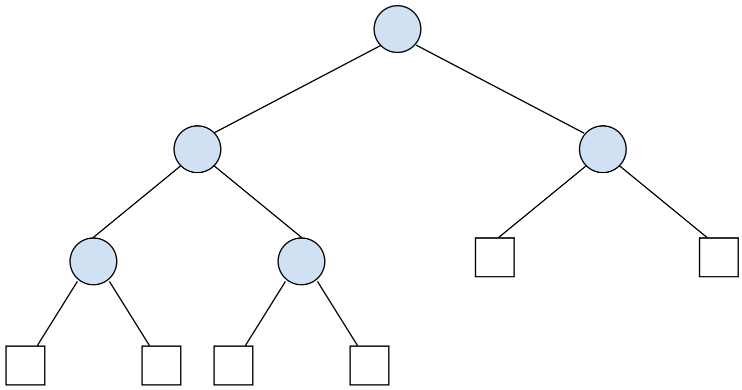
Example



BINARY_TREE Operations

- *IsRoot*: W × BT → B :
  The function value *IsRoot*(*w*, *T*) is *true* if the window *w* is over the root; otherwise it is *false*

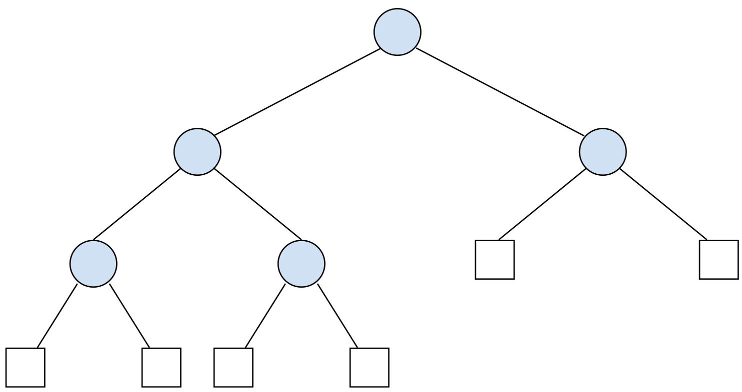
Example



BINARY_TREE Operations

- *IsExternal*: W × BT → B :
  The function value *IsExternal*(*w*, *T*) is *true* if the window *w* is over an external node of *T*; otherwise it is *false*

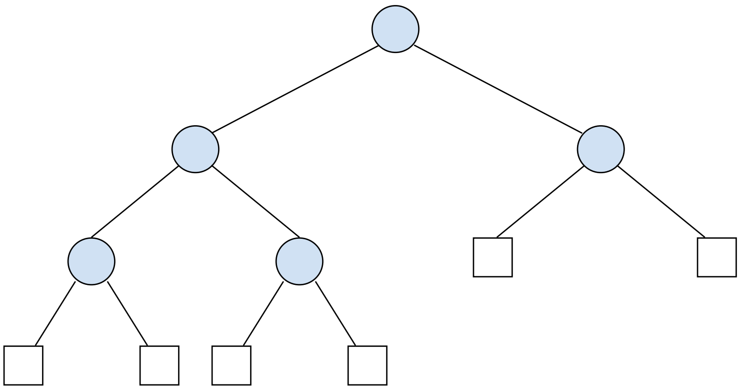
Example



BINARY_TREE Operations

- *Child*: N × W × BT → W :
  The function value *Child*(*i*, *w*, *T*) is undefined if the node in the window *W* is external or the node in *w* is internal and *i* is neither 1 nor 2; otherwise it is the *i*th child of the node in *w*

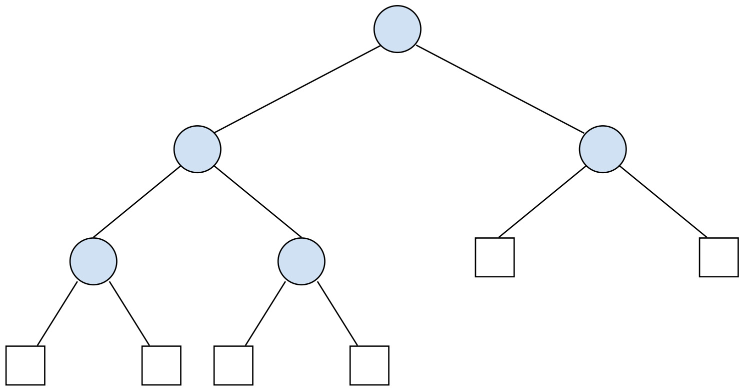
Example



BINARY_TREE Operations

- *Parent*: W $\times$ BT $\to$ W :
  The function value *Parent*(*w*, *T*) is undefined if *T* is empty or *w* is over the root of *T*; otherwise it is the window position of the parent of the node in the window *w*

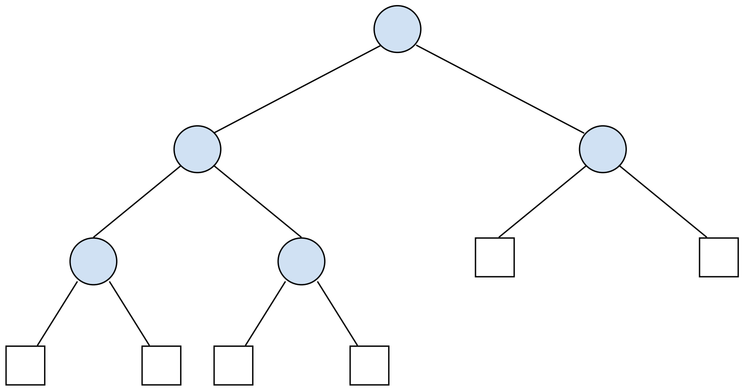
Example



BINARY_TREE Operations

- *Examine*: W × BT → *I* :
  The function value *Examine*(*w*, *T*) is undefined if *w* is over an external node; otherwise it is element at the internal node in the window *w*

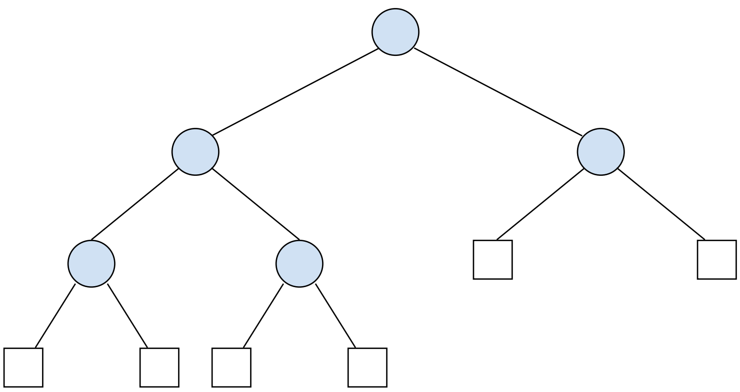
Example



BINARY_TREE Operations

- *Replace*: $E \times$ W $\times$ BT $\rightarrow$ BT :
  The function value *Replace*(*e*, *w*, *T*) is undefined if *w* is over an external node; otherwise it is *T*, with the element at the internal node in *w* replaced by *e*

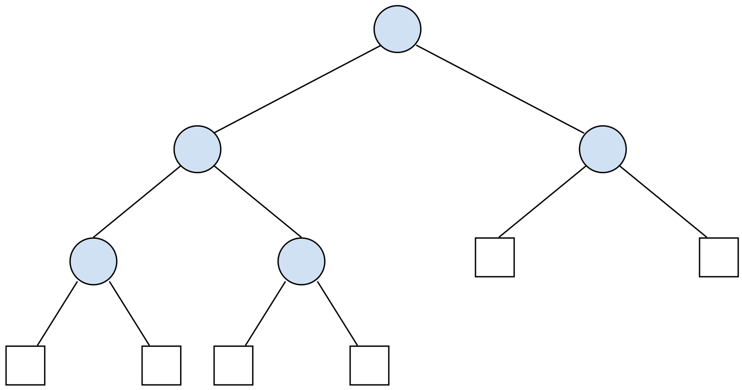
Example



BINARY_TREE Operations

- *Insert*: $E \times W \times BT \rightarrow W \times BT$:
  The function value *Insert*(*e*, *w*, *T*) is undefined if *w* is over an internal node; otherwise it is *T*, with the external node in *w* replaced by a new internal node with two external children.
  - Furthermore, the new internal node is given the value *e* and the window is moved over the new internal node.

Example



BINARY_TREE Operations

- *Delete*: W × BT → W × BT :

  – The function value *Delete*(*w*, *T*) is undefined if *w* is over an external node;

  – If w is over a leaf node (both its children are external nodes), then the function value is T with the internal node to be deleted replaced by its left external node

BINARY_TREE Operations

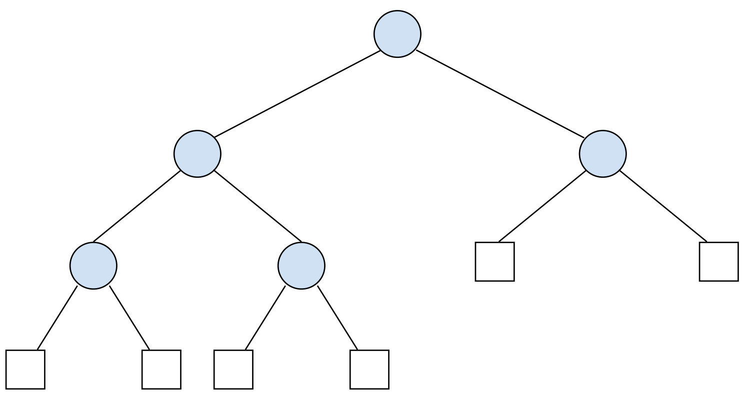
- *Delete*: W × BT → W × BT :

  If w is over an internal node with just one internal node child, then the function value is T with the internal node to be deleted replaced by its child

BINARY_TREE Operations

- *Delete*: W × BT → W × BT :
  - if w is over an internal node with two internal node children, then the function value is T with the internal node to be deleted replaced by the leftmost internal node descendent in its right sub-tree
  - In all cases, the window is moved over the replacement node.

Example

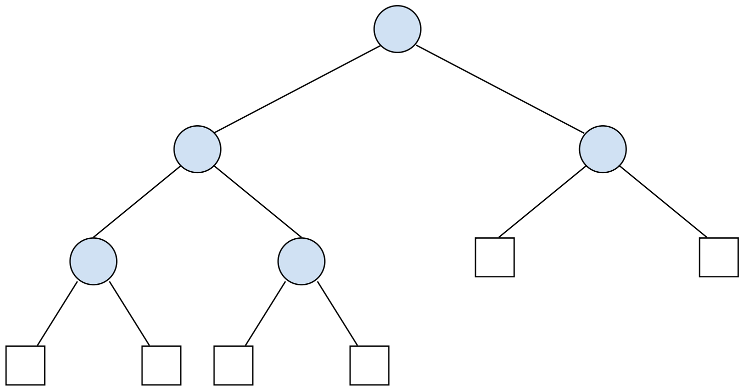


BINARY_TREE Operations

- *Left*: W × BT → W :

  The function value *Left*(*w*, *T*) is undefined if *w* is over an external node; otherwise it is the window position of the left (or first) child of the node *w*

Example

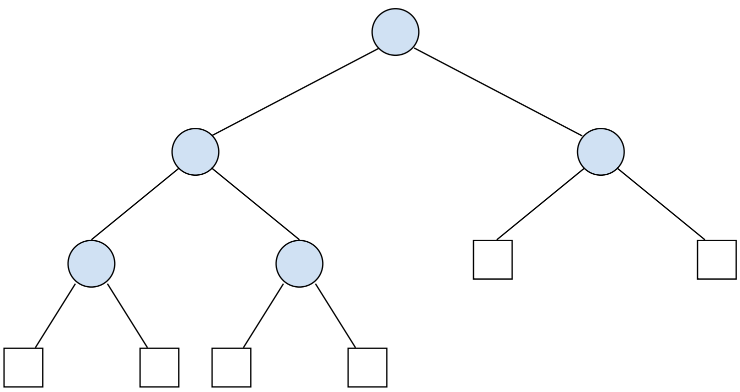


BINARY_TREE Operations

- *Right*: W × BT → W :

  The function value *Right*(*w*, *T*) is undefined if *w* is over an external node; otherwise it is the window position of the right (or second) child of the node *w*

Example

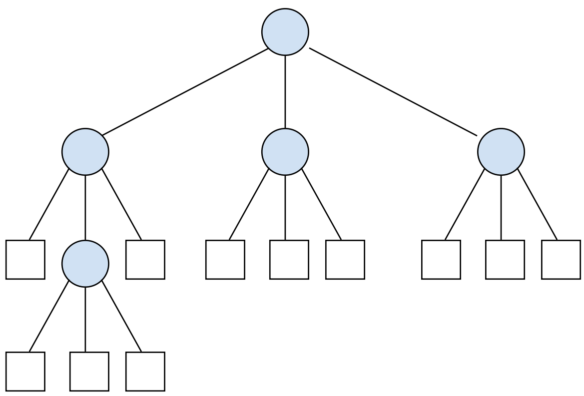


TREE Operations

- *Degree*: W × T → *I* :

  The function value *Degree*(*w*, *T*) is the degree of the node in the window *w*

d-ary Tree

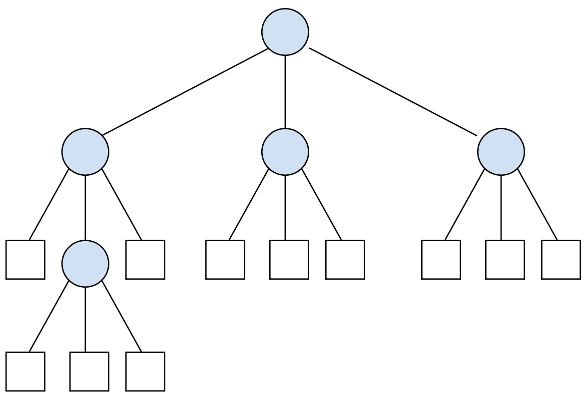


TREE Operations

- *Child*: N × W × T → W :

  The function value *Child*(i, w, T) is undefined if the node in the window w is external, or if the node in w is internal and i is outside the range 1..d, where d is the degree of the node; otherwise it is the ith child of the node in w

d-ary Tree



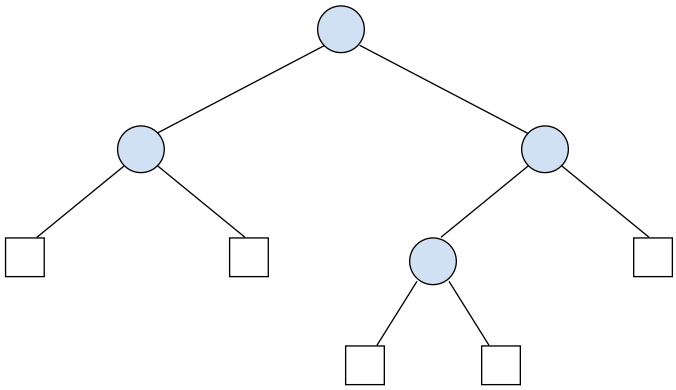
BINARY_TREE Representation

```
/* pointer implementation of BINART_TREE ADT */

#include <stdio.h>
#include <math.h>
#include <string.h>

#define FALSE 0
#define TRUE 1

typedef struct {
           int number;
           char *string;
        } ELEMENT_TYPE;
```

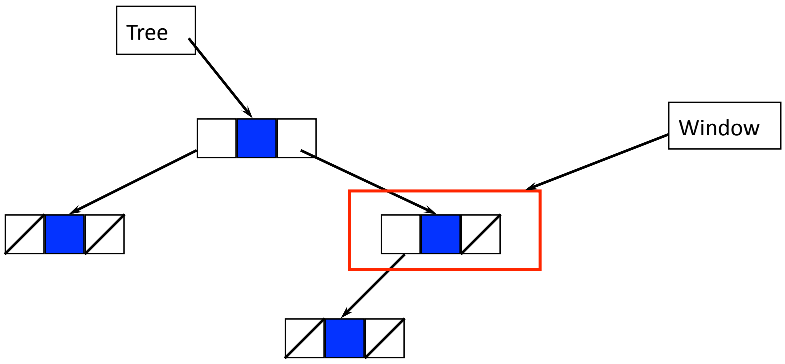

BINARY_TREE Representation

```
typedef struct node *NODE_TYPE;

typedef struct node{
           ELEMENT_TYPE element;
           NODE_TYPE left, right;
        } NODE;

typedef NODE_TYPE BINARY_TREE_TYPE;
typedef NODE_TYPE WINDOW_TYPE;
```


BINARY_TREE Representation



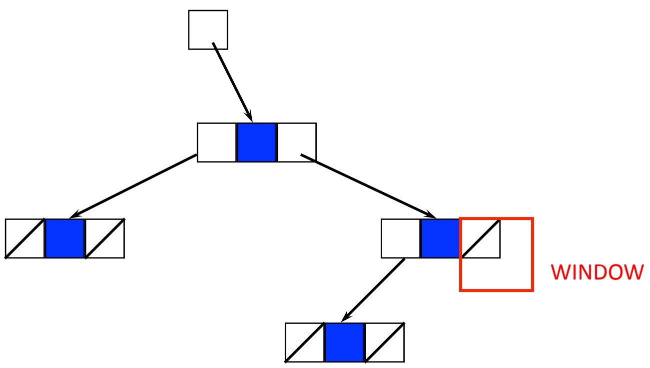
BINARY_TREE Representation



BINARY_TREE Representation

- This implementation assumes that we are going to represent external nodes as NULL links
- For many ADT operations, we need to know if the window is over an internal or an external node
  - we are over an external node if the window is NULL

BINARY_TREE Representation



BINARY_TREE Representations

- Whenever we insert an internal node (remember we can only do this if the window is over an external node) we simply make its two children NULL

