Data Structures and Algorithms for Engineers

Module 6: Trees

Lecture 4: Height Balanced Trees: Red-Black Trees

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Red-Black Trees

- The goal of height-balancing is to ensure that the tree is as complete as possible and that, consequently, it has minimal height for the number of nodes in the tree

- As a result, the number of probes it takes to search the tree (and the time it takes) is minimized.

Red-Black Trees

- A perfect or a complete tree with n nodes has height $O(\log_2 n)$
  - So, the time it takes to search a perfect or a complete tree with n nodes is $O(\log_2 n)$
- A skinny tree could have height $O(n)$
  - So, the time it takes to search a skinny tree can be $O(n)$
- Red-Black trees are similar to AVL trees in that they allow us to construct trees which have a guaranteed search time $O(\log_2 n)$

Red-Black Trees

A red-black tree is a binary tree whose nodes can be coloured either red or black to satisfy the following three conditions:

1. **Black condition**:
   Each root-to-frontier path contains exactly the same number of black nodes

2. **Red condition**:
   Each red node that is not the root has a black parent

3. Each external node is **black**

Red-Black Trees

- A red-black search tree is a red-black tree that is also a binary search tree

- For all n >= 1, every red-black tree of size n has height $O(\log_2 n)$

  – Thus, red-black trees provide a guaranteed worst-case search time of $O(\log_2 n)$

Red-Black Trees

Red-black tree (condition 3)

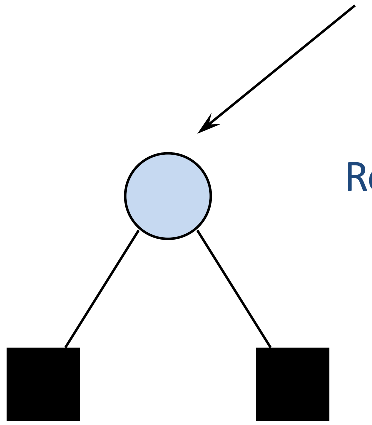
1. **Black condition**:
   Each root-to-frontier path contains exactly the same number of black nodes

2. **Red condition**:
   Each red node that is not the root has a black parent

3. Each external node is **black**

Red-Black Trees

Undetermined colour (both red and black satisfy conditions 1 & 2)



Red-black tree (condition 3)

1. **Black condition**:
   Each root-to-frontier path contains exactly the same number of black nodes
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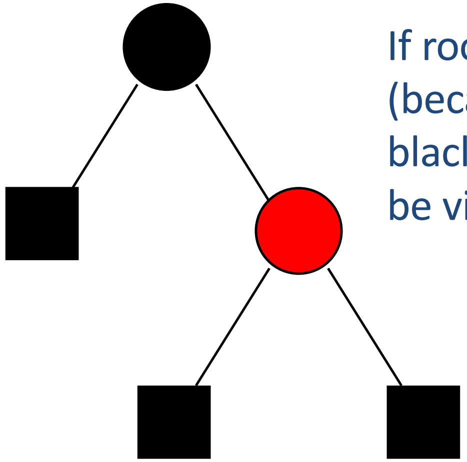
Red-Black Trees



If root was red, then right child would have to be black (because if it was red, by Condition 2 it would have to have a black parent) but then Condition 1, the black condition, would be violated … so the root can't be red in this case.

1. **Black condition**:
   Each root-to-frontier path contains exactly the same number of black nodes

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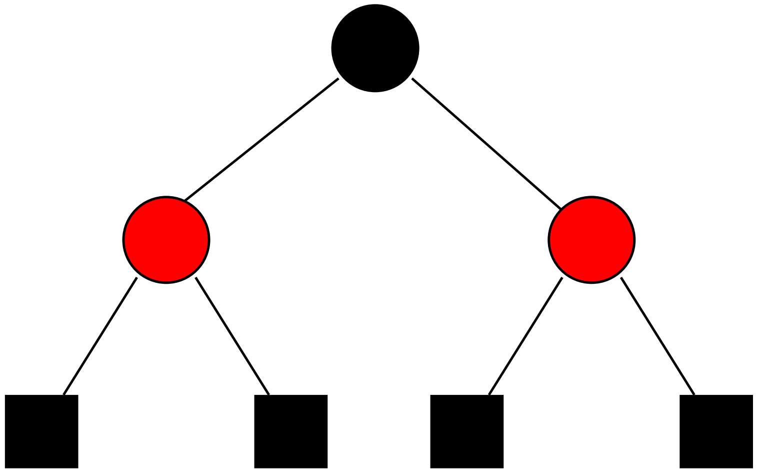
   Each red node that is not the root has a black parent

3. Each external node is **black**

Red-Black Trees



1. **Black condition**:
   Each root-to-frontier path contains exactly the same number of black nodes

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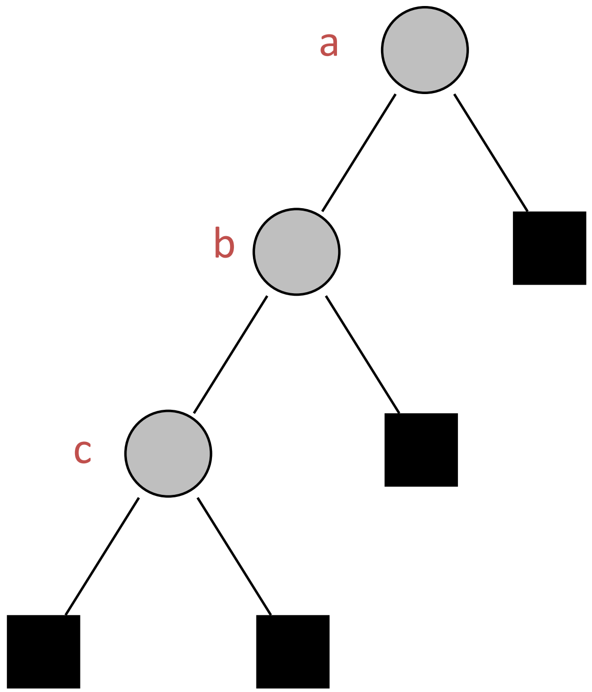
Red-Black Trees



To satisfy black condition, either

(1) node a is black and nodes b and c are red, or

(2) nodes a, b, and c are red

In both cases, a red condition is violated

Therefore, this is not a red-black tree (i.e., it cannot be coloured in a way that satisfies all three conditions)

1. **Black condition**:
   Each root-to-frontier path contains exactly the same number of black nodes
2. **Red condition**:
   Each red node that is not the root has a black parent
3. Each external node is **black**

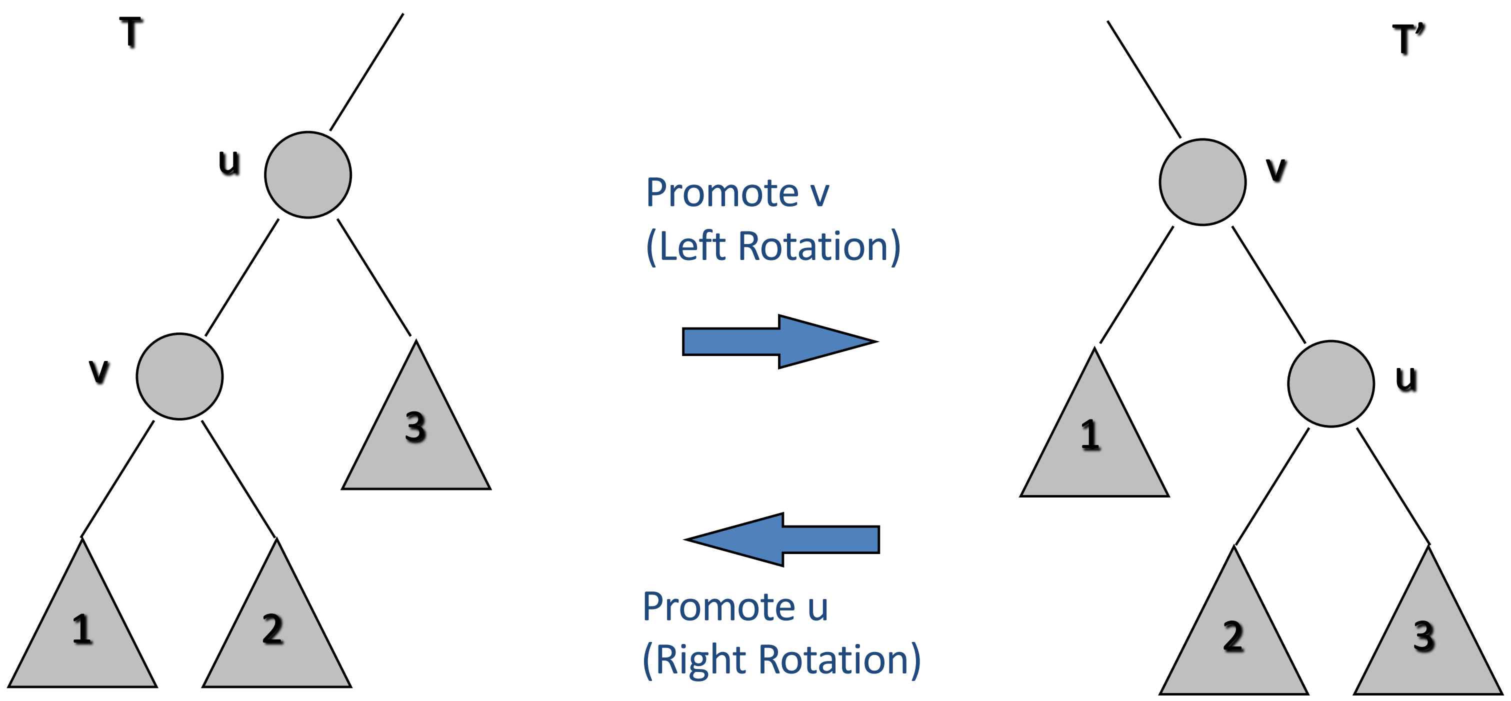
Red-Black Trees

- Insertions and deletions can cause red and black conditions to be violated
- Trees then have to be restructured
- Restructuring called a promotion (or rotation)
  - Single promotion
  - 2 promotion

Red-Black Trees

- Single promotion
- Also referred to as
  - single (left) rotation
  - single (right) rotation
- Promotes a node one level

Red-Black Trees

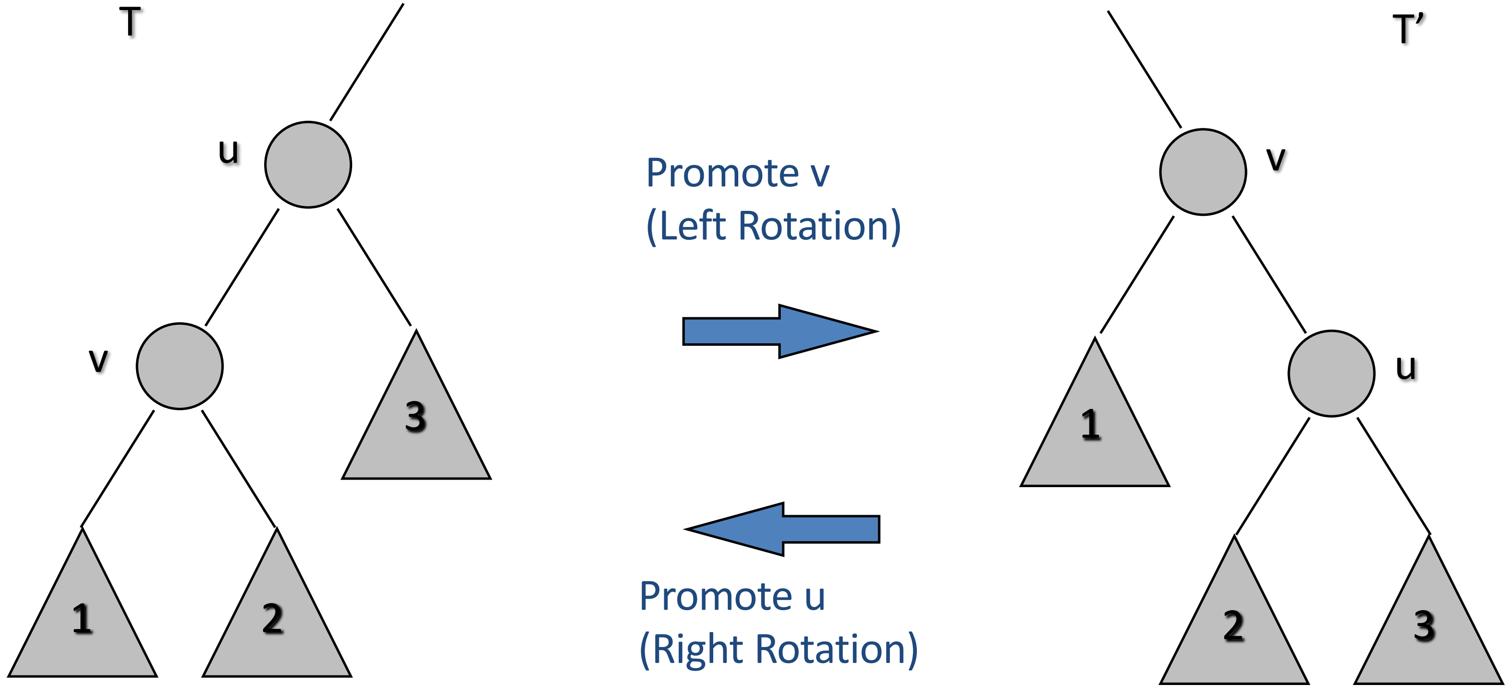


Red-Black Trees

- A single promotion (Left Rotation or Right Rotation) preserves the binary-search condition

- Same manner as an AVL rotation

Red-Black Trees



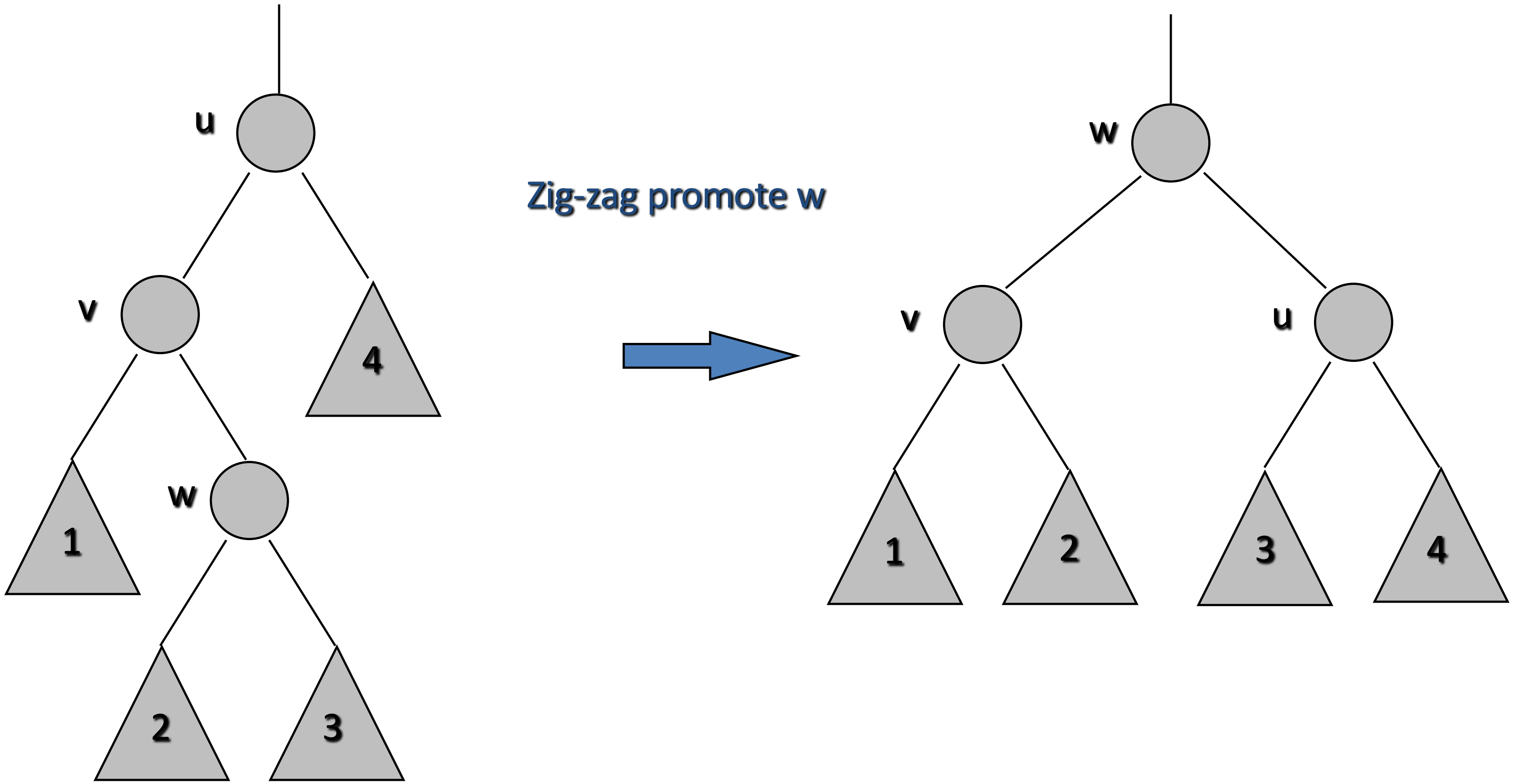
keys(1) < key(v) < key(u)
key(v) < keys(2) < key(u)
key(u) < keys(3)

keys(1) < key(v)
key(v) < keys(2) < key(u)
key(v) < key(u) < keys(3)

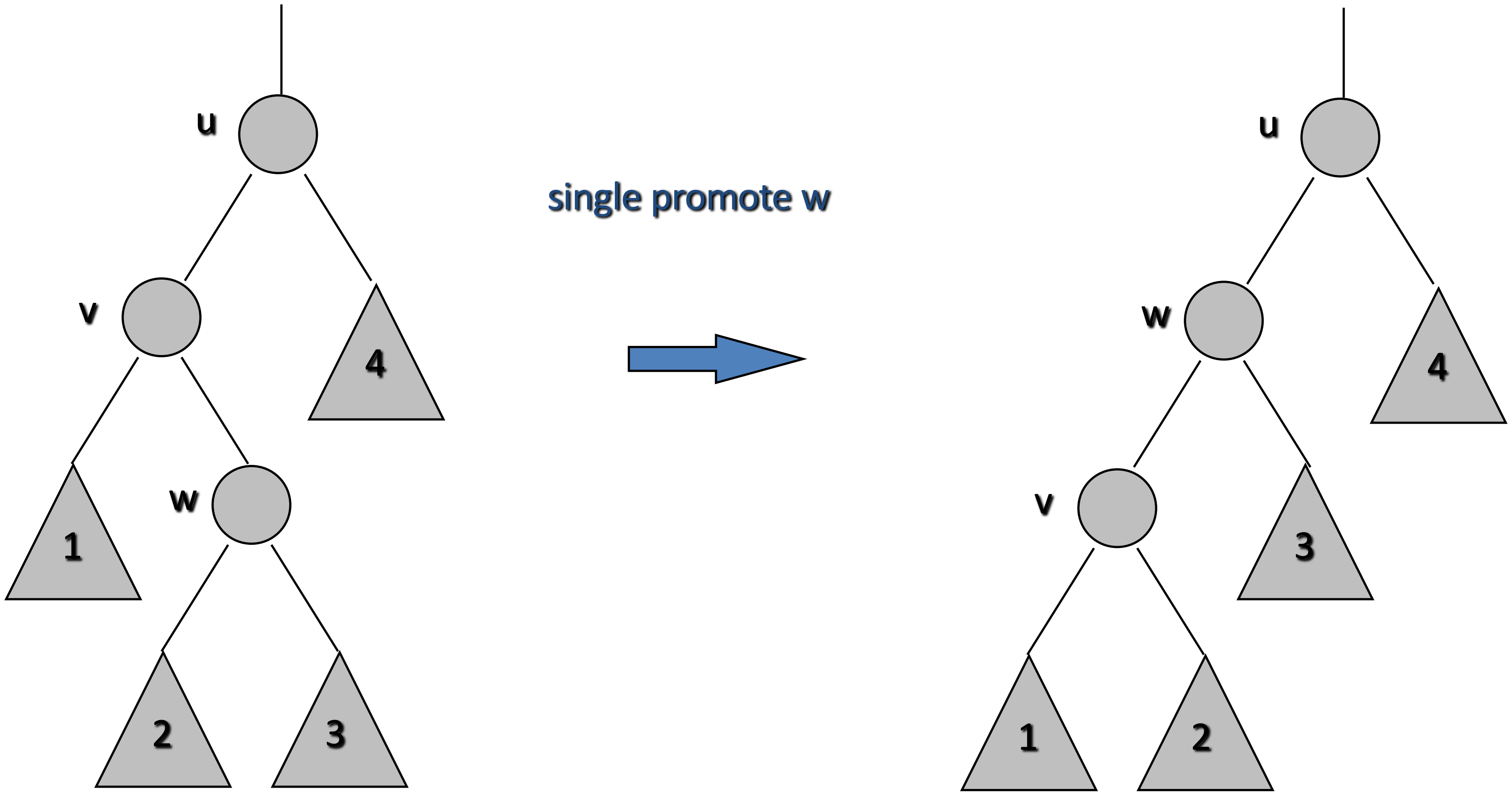
Red-Black Trees

- 2-Promotion
- Zig-zag promotion
- Composed of two single promotions
- And hence preserves the binary-search condition

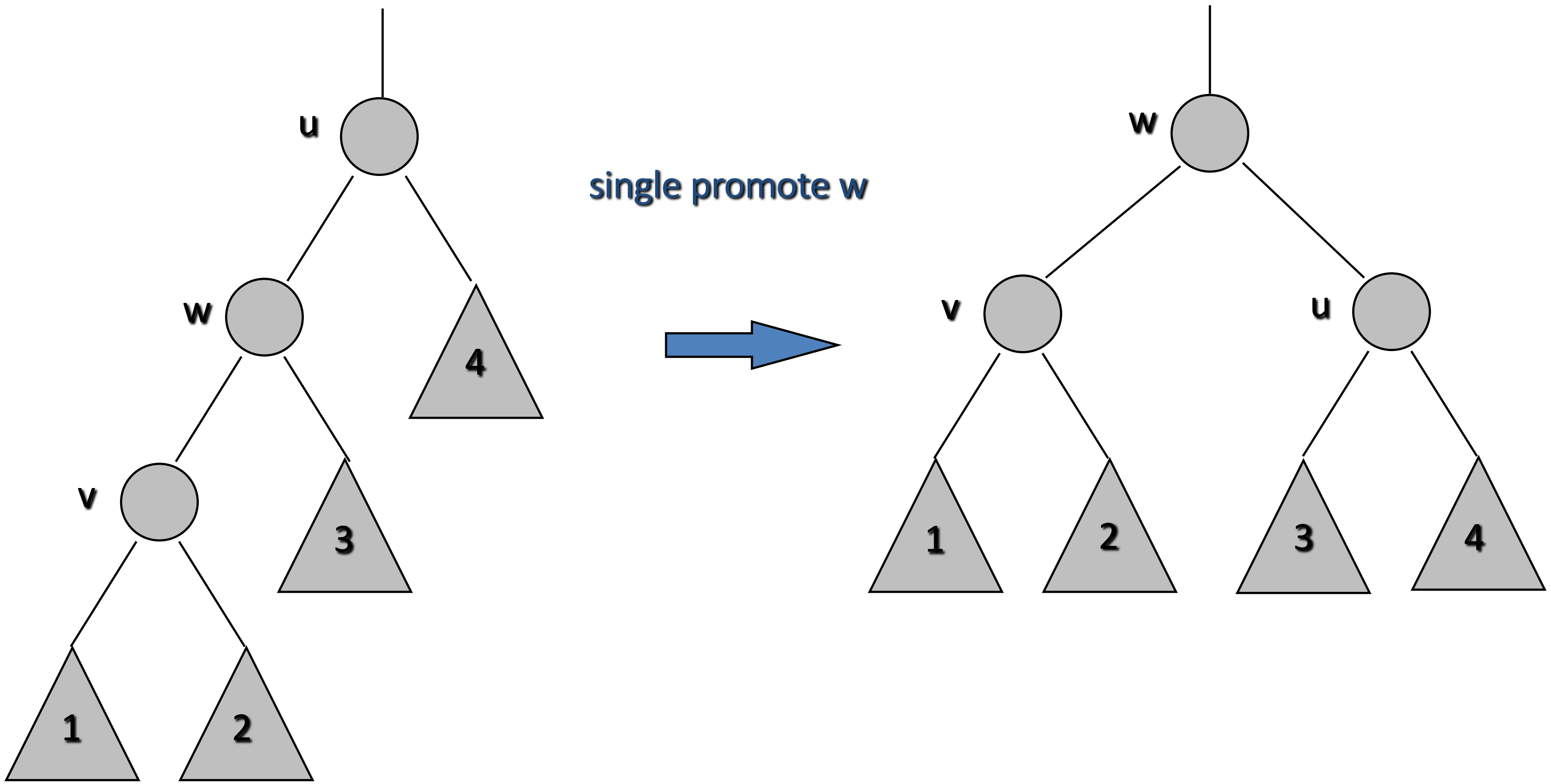
Red-Black Trees



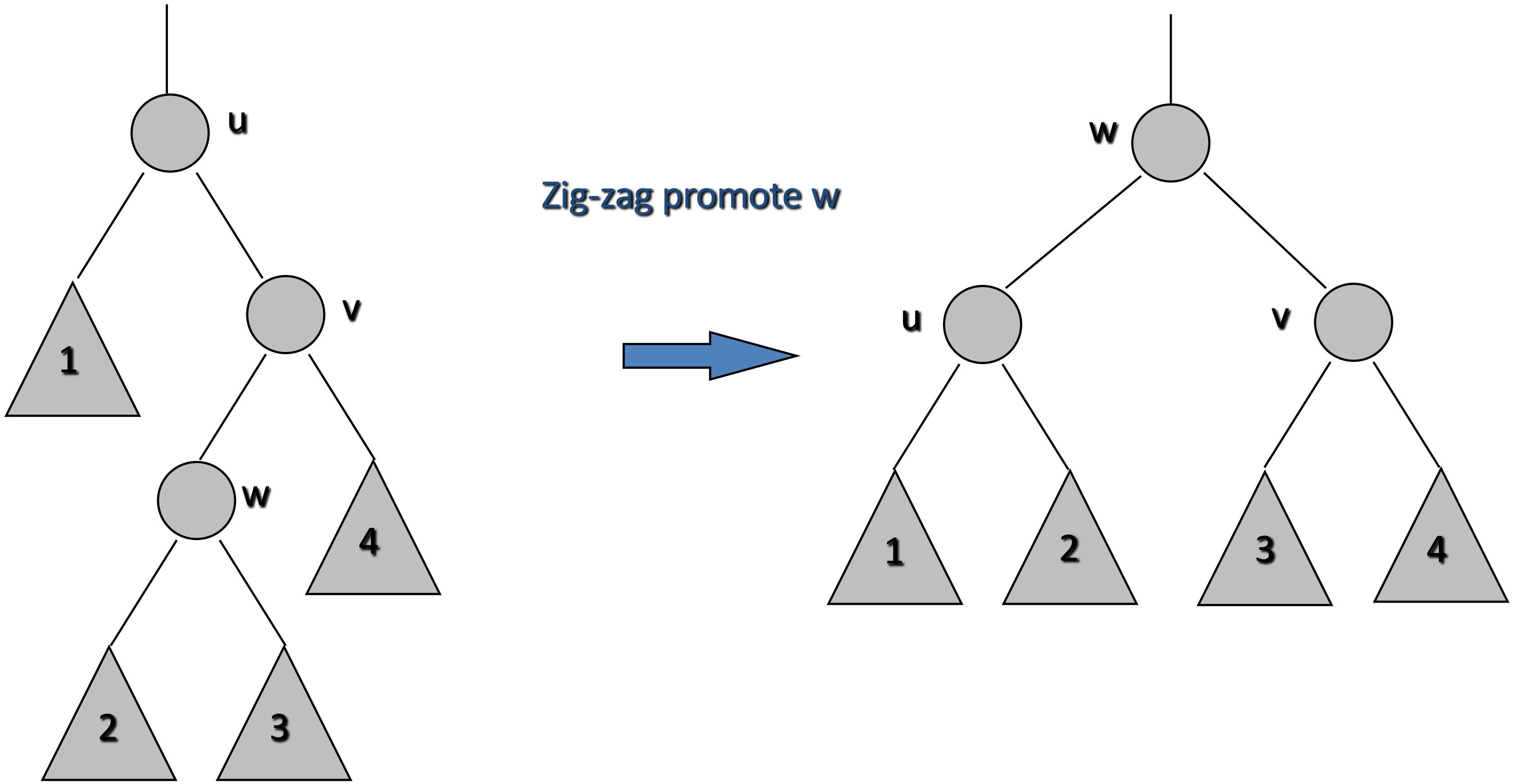
Red-Black Trees



Red-Black Trees



Red-Black Trees



Red-Black Trees

Insertions

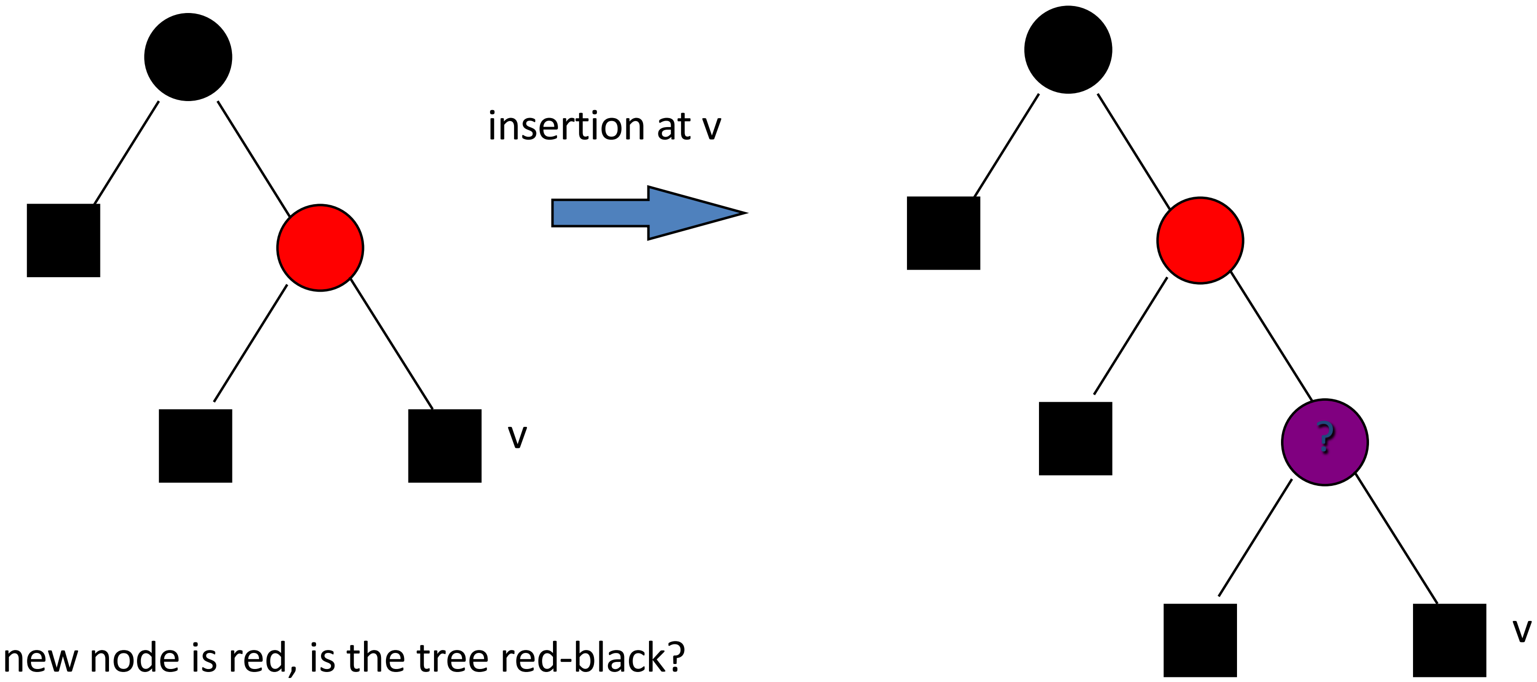
- A red-black tree can be searched in logarithmic time, worst case
- Insertions may violate the red-black conditions necessitating restructuring
- This restructuring can also be effected in logarithmic time
- Thus, an insertion (or a deletion) can be effected in logarithmic time

Red-Black Trees

- Just as with AVL trees, we perform the insertion by
  - first searching the tree until an external node is reached (if the key is not already in the tree)
  - then inserting the new (internal) node
- We then have to recolour and restructure, if necessary

Red-Black Trees

1. **Black condition**:
   Each root-to-frontier path contains exactly the same number of black nodes
2. **Red condition**:
   Each red node that is not the root has a black parent
3. Each external node is **black**

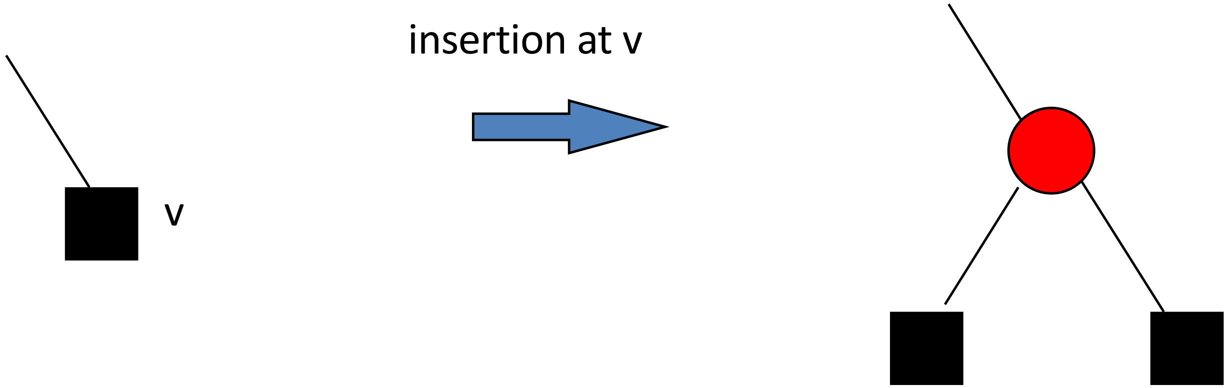


If new node is red, is the tree red-black?
If the new node is black, is the tree red-black?

Red-Black Trees

- Recolouring:
  - Colour new node red
  - This preserves the black condition
  - but may violate the red condition
- Red condition can be violated only if the parent of an internal node is also red
- Must transform this ‘almost red-black tree’ into a red-black tree

Red-Black Trees



Red-Black Trees

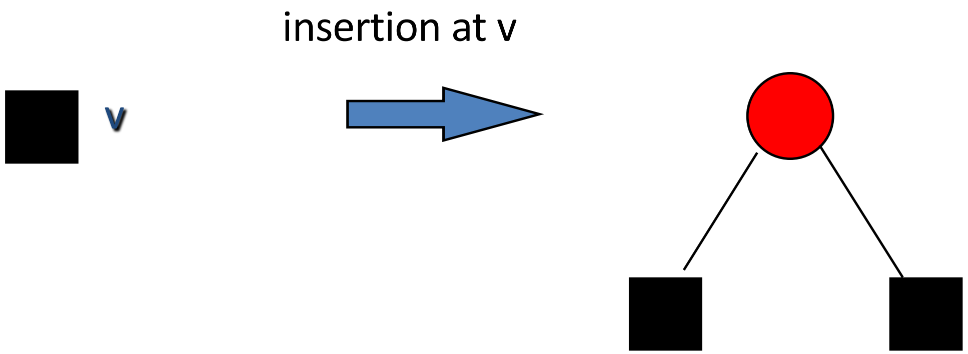
Recolouring and restructuring algorithm

- The node u is a red node in a BST, T
- u is the only candidate violating node
- Apart from u, the tree T is red-black

Red-Black Trees

Case 1:

- u is the root
- T is red-black

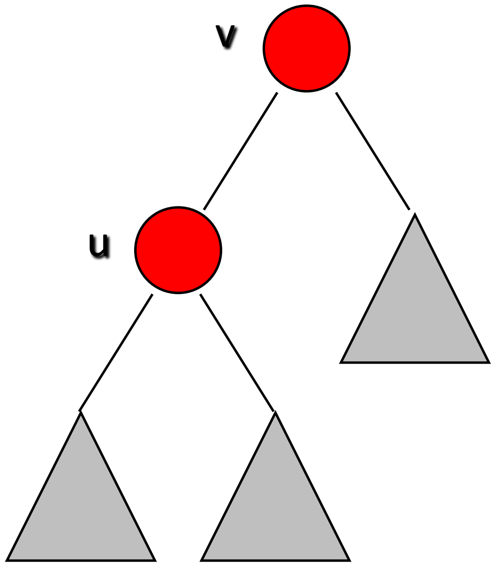


Red-Black Trees

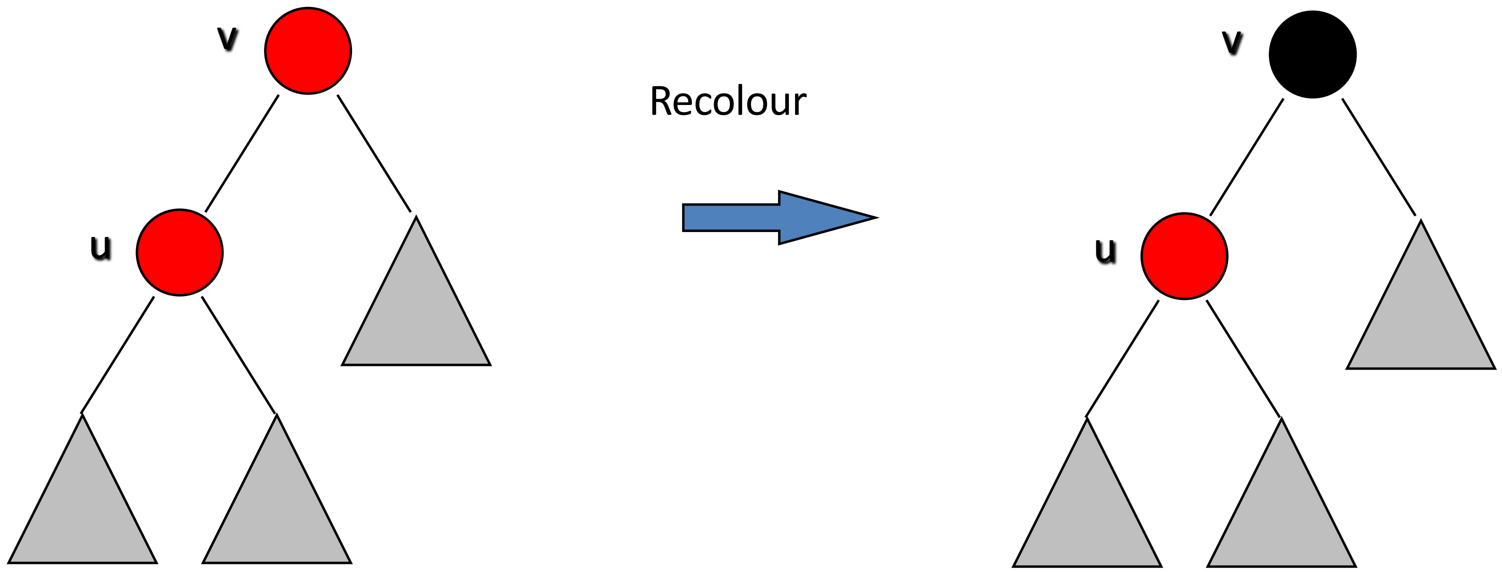
Case 2:

- u is not the root
- its parent v is the root
- Colour v black

  - Since v is the parent and the root,
    it is on the path to all external nodes
    and, therefore, the black condition is satisfied

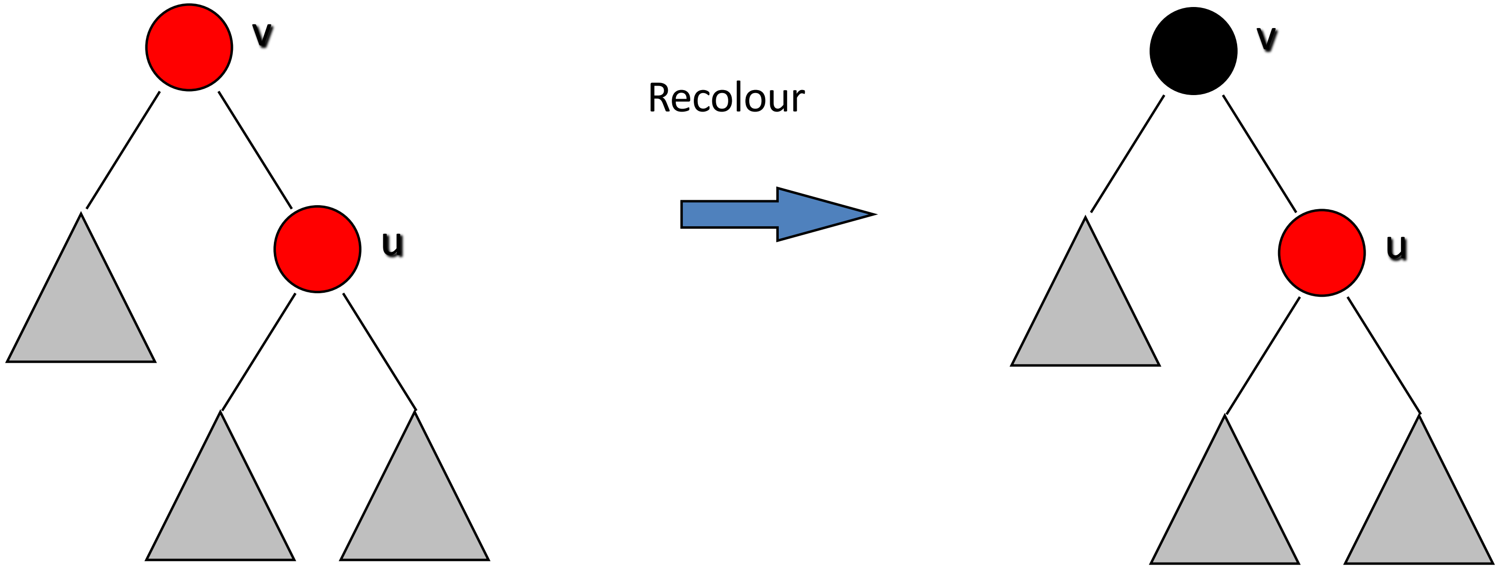


Red-Black Trees



Is there anything unexpected about this figure?

Red-Black Trees

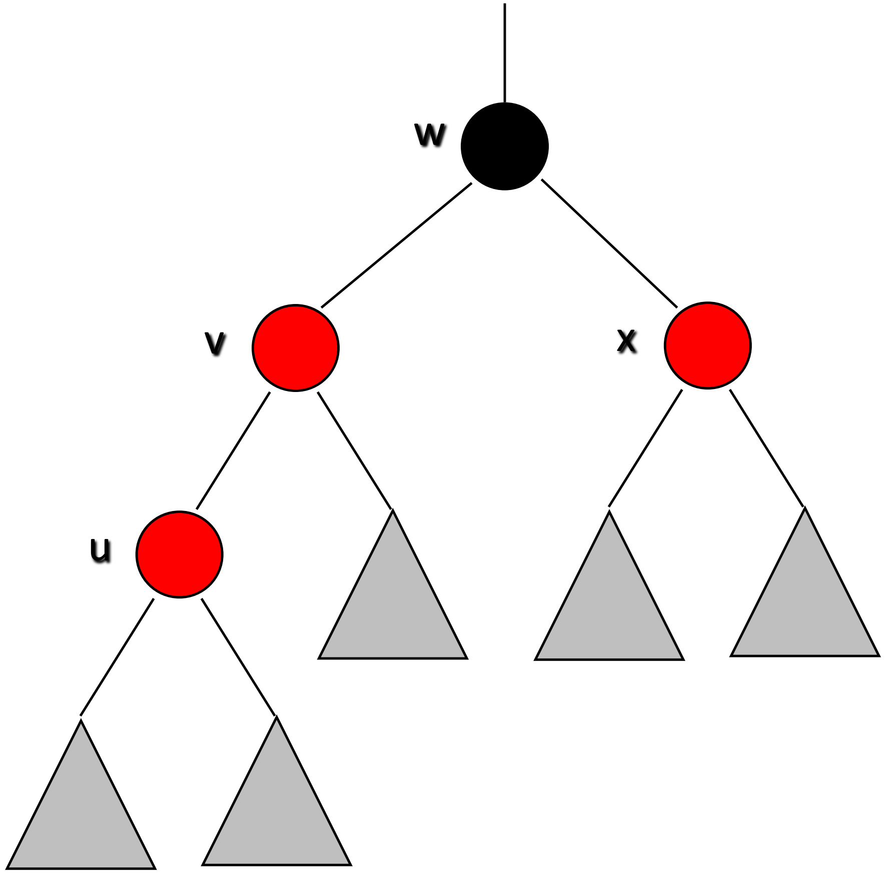


Is there anything unexpected about this figure?

Red-Black Trees

Case 3:

- u is not the root,
- its parent v is not the root,
- v is the left child of its parent w
- (x is the right child of w,
  i.e., x is v's sibling)



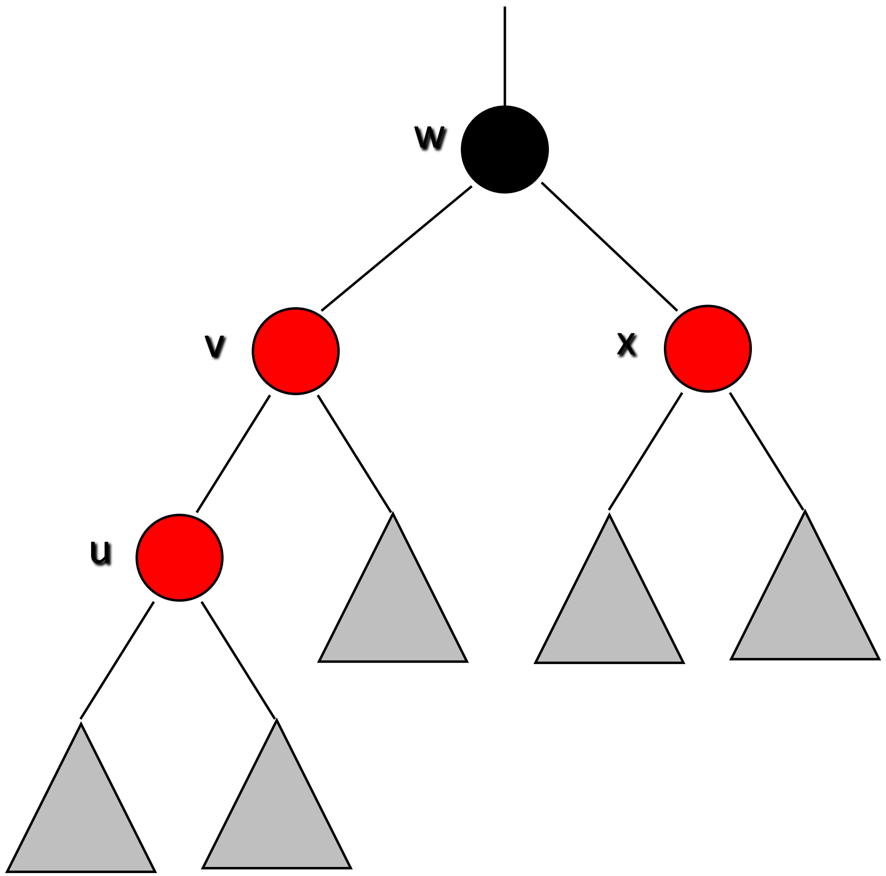
Red-Black Trees

Case 3.1:

- x is red
- Colour v and x black and w red
- Now repeat the restructuring with u := w

  (since the recolouring of w to red may cause a red violation)

Red-Black Trees



Note:
w must be black,
v must be red,
u must be red.
Why?

Recolour

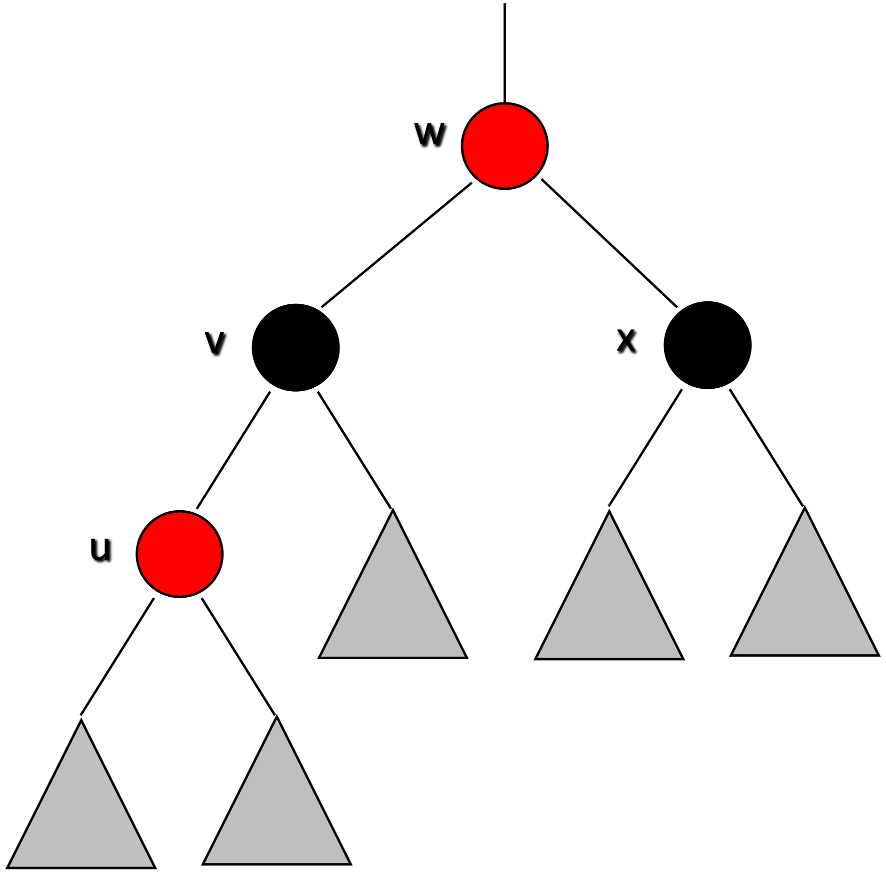
Red-Black Trees

- u must be red because we colour new nodes that way by convention (to preserve the black condition)

- v must be red because otherwise it would be black and then we wouldn't have violated the red condition and we wouldn't be restructuring anything!

- w must be black because every red node (that isn't the root) has a black parent (and x is red so w must be black)

Red-Black Trees

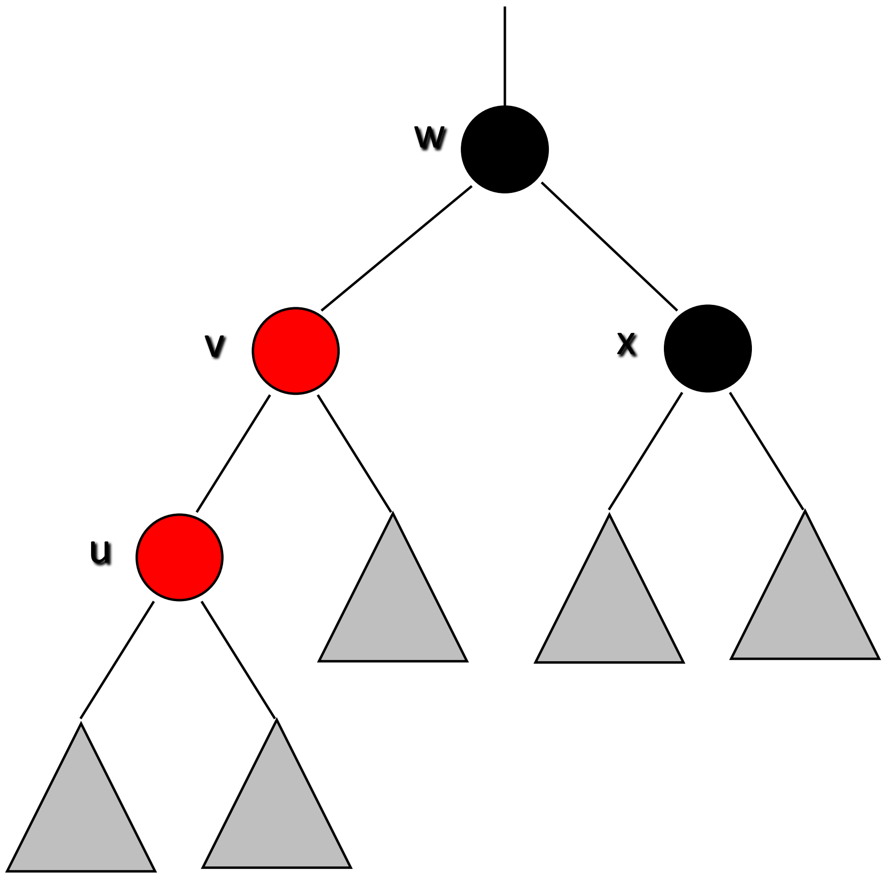


Red-Black Trees

Case 3.2:

- x is black
- u is the left child of v

Red-Black Trees

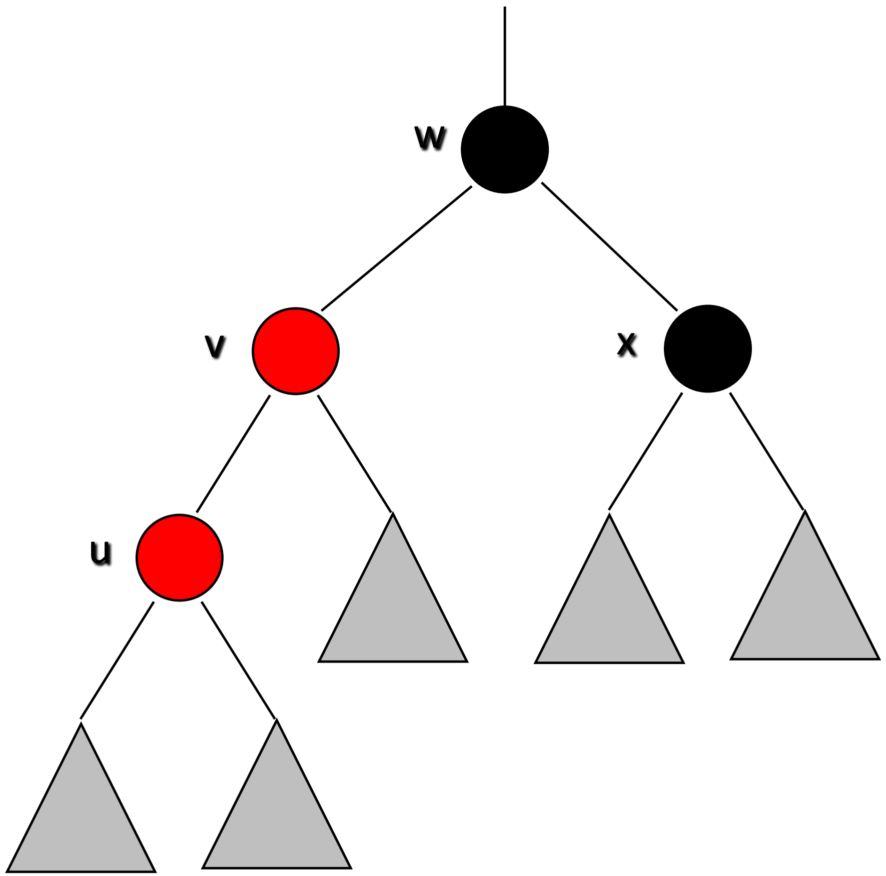


Red-Black Trees

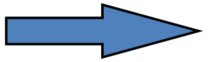
Case 3.2:

- x is black
- u is the left child of v
- Promote v
- Colour v black
- Colour w red

Red-Black Trees

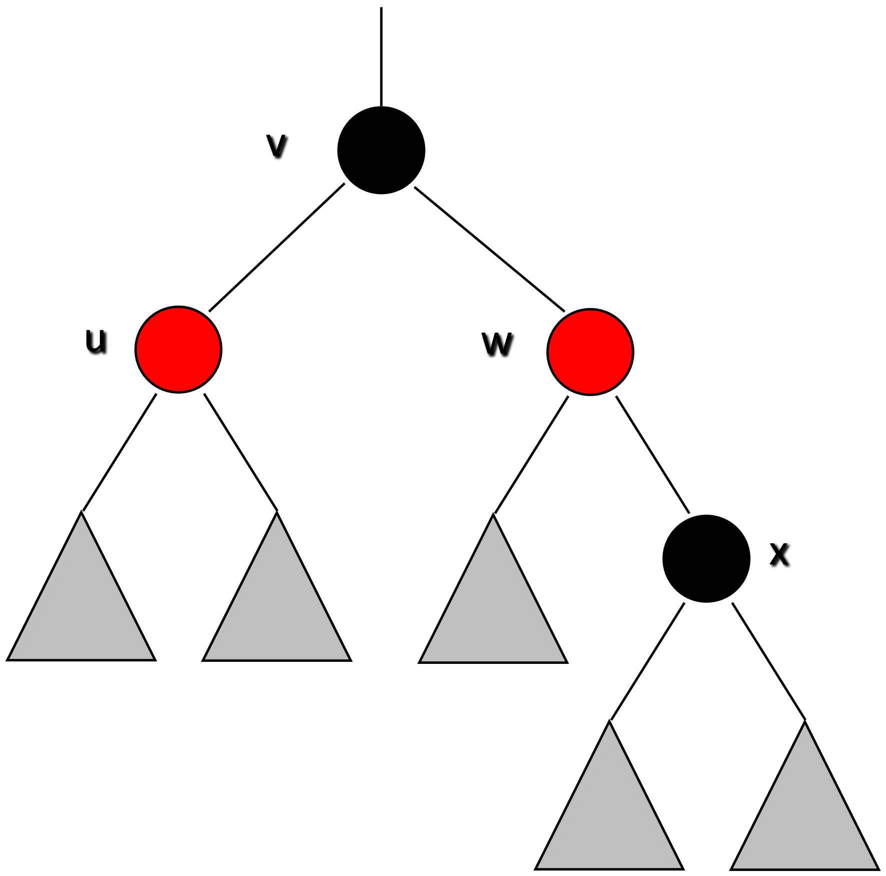


Restructure and recolour



Promote v;
colour v black;
colour w red

Red-Black Trees

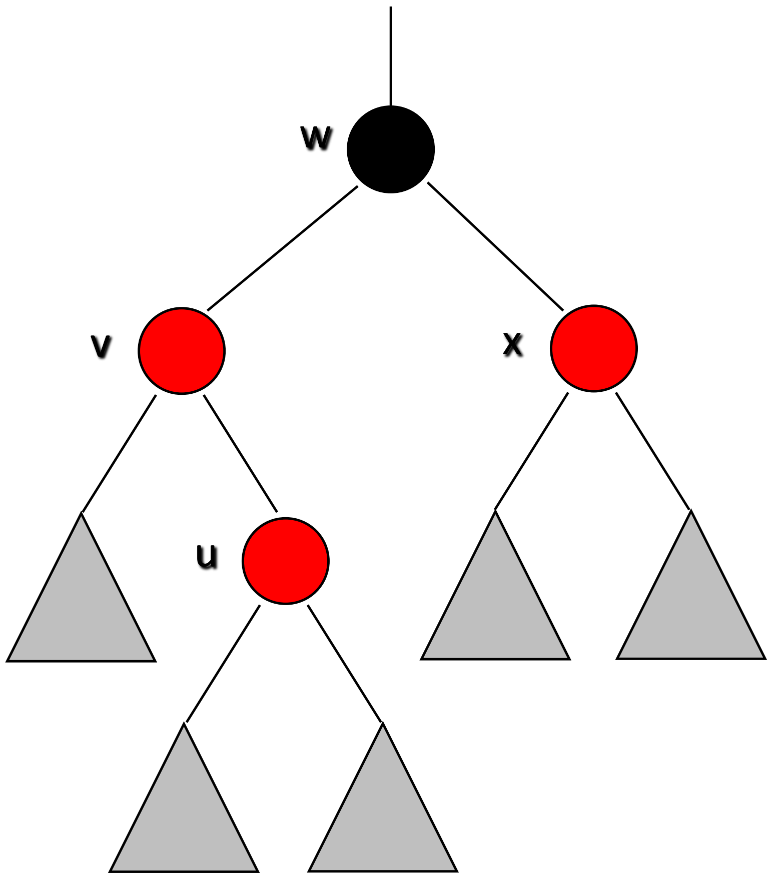


Red-Black Trees

Case 3.3:

- x is red
- u is the right child of v

Red-Black Trees



Red-Black Trees

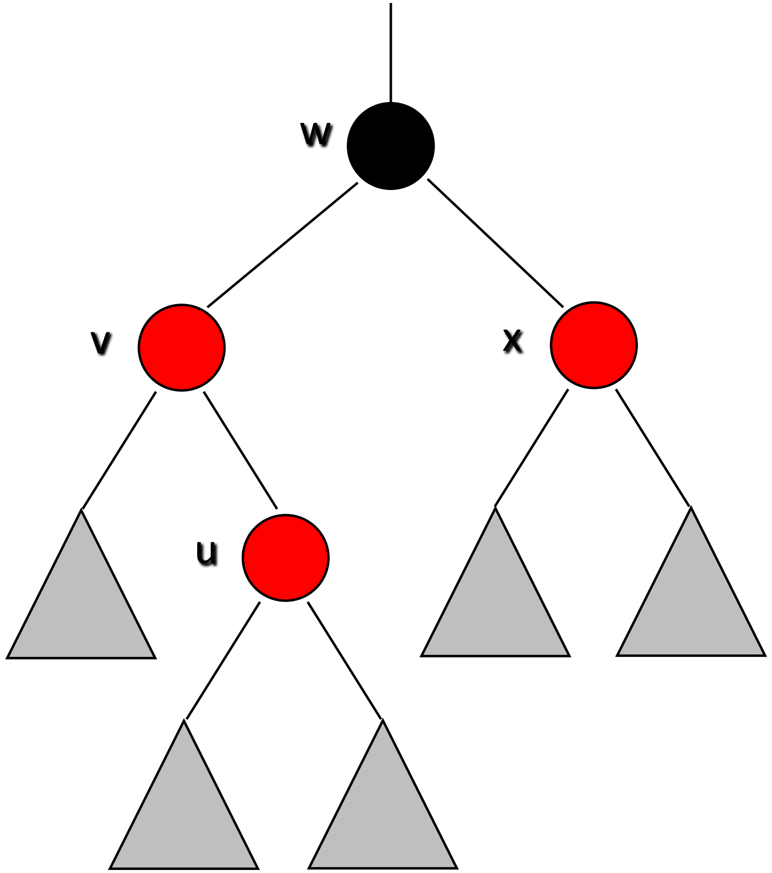
Case 3.3:

- x is red
- u is the right child of v
- Colour v and x black
- Colour w red

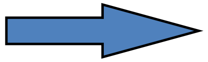
- Repeat the restructuring with u := w

  (since the recolouring of w to red may cause a red violation)

Red-Black Trees

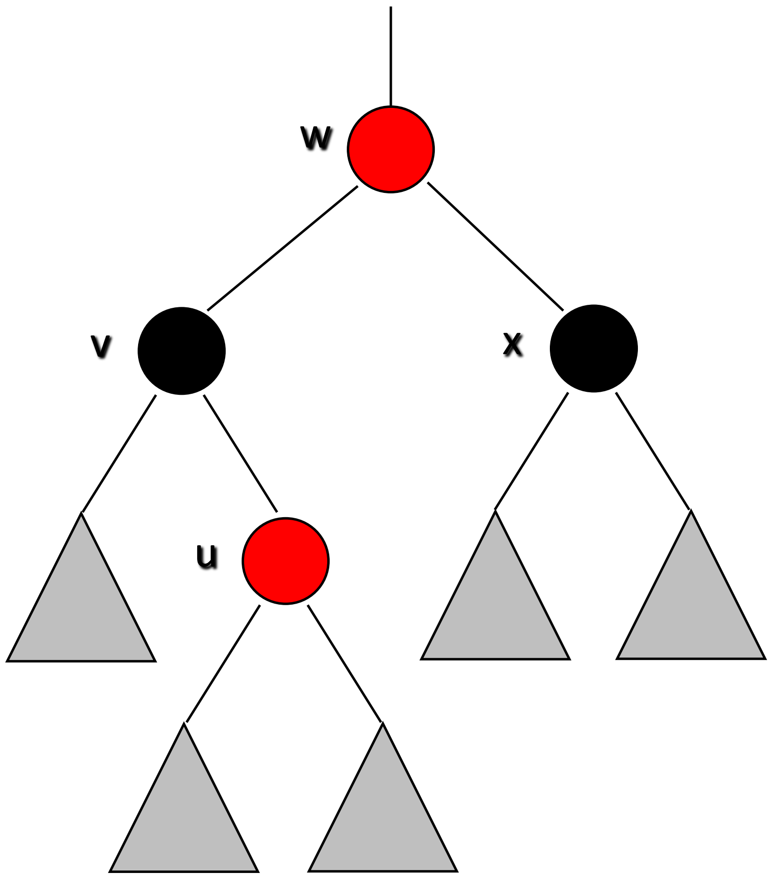


Recolour



Colour v and x black
Colour w red

Red-Black Trees

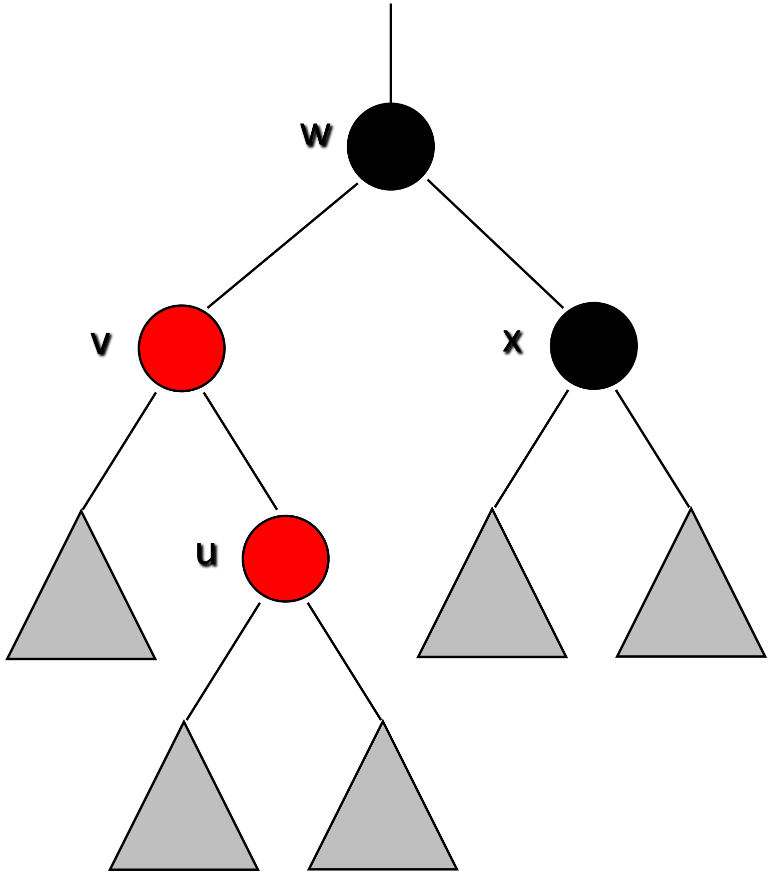


Red-Black Trees

Case 3.4:

- x is black
- u is the right child of v

Red-Black Trees

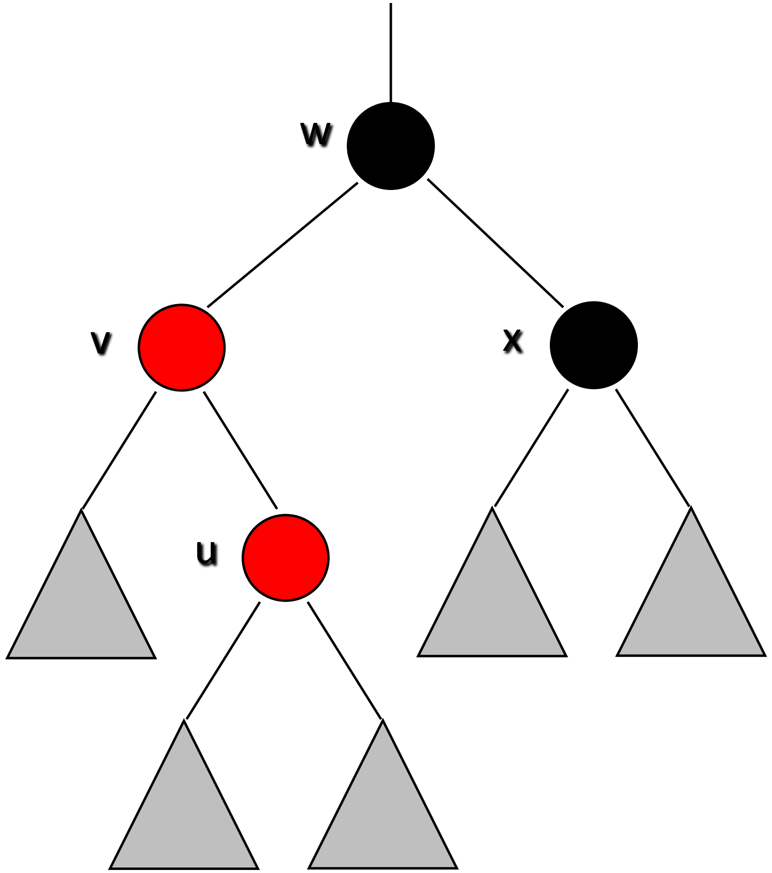


Red-Black Trees

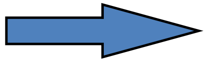
Case 3.4:

- x is black
- u is the right child of v
- Zig-zag promote u
- Colour u black
- Colour w red

Red-Black Trees

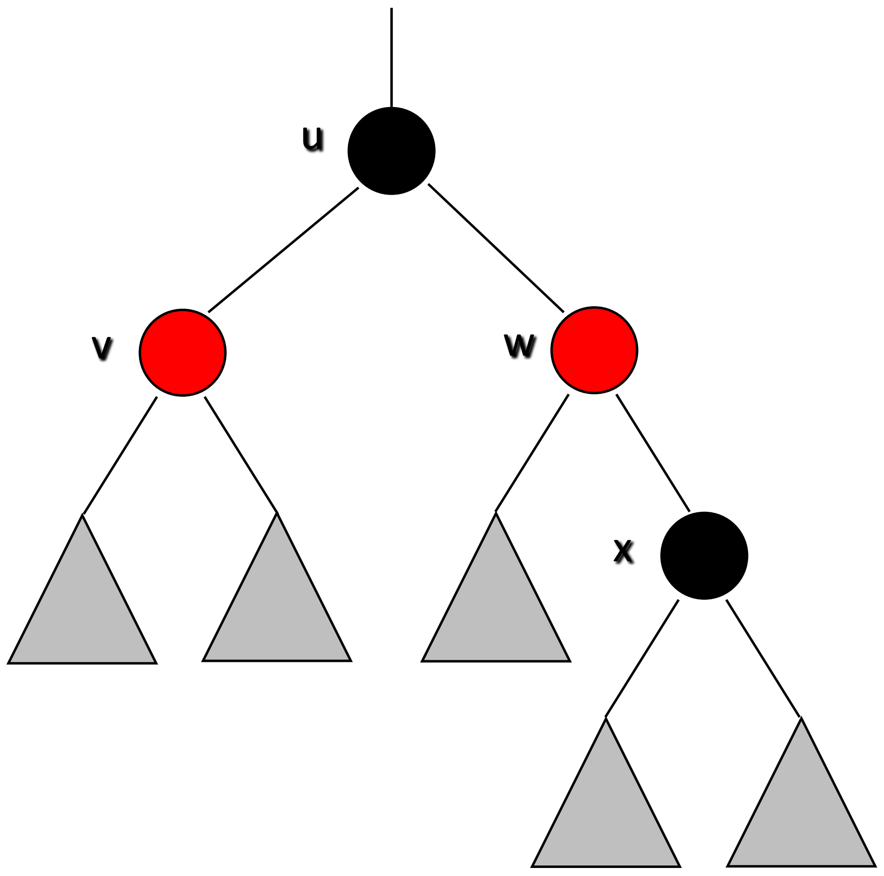


Restructure and recolour



Zig-zag promote u;
colour u black;
colour w red

Red-Black Trees



Red-Black Trees

- Case 4:

  - u is not the root
  - its parent v is not the root
  - v is the right child of its parent w
  - (x is the left child of w, i.e., x is v's sibling)

- This case is symmetric to case 3