Data Structures and Algorithms for Engineers

Module 6: Trees

Lecture 5: Optimal Code Trees. Huffman's Algorithm.

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Optimal Code Trees

- First application: coding and data compression

- We will define optimal variable-length binary codes and code trees

- We will study Huffman's algorithm which constructs them

- Huffman's algorithm is an example of a Greedy Algorithm, an important class of simple optimization algorithms

Text, Codes, and Compression

- Computer systems represent data as bit strings
- Encoding: transformation of data into bit strings
- Decoding: transformation of bit strings into data
- The code defines the transformation

Text, Codes, and Compression

- For example: ASCII, the international coding standard, uses a 7-bit code
- HEX Code – Character
- 20 - <space>
- 41 – A
- 42 – B
- 61 - a

Text, Codes, and Compression

- Such encodings are called
  - fixed-length or
  - block codes
- They are attractive because the encoding and decoding is extremely simple
  - For coding, we can use a block of integers or codewords indexed by characters
  - For decoding, we can use a block of characters indexed by codewords

Text, Codes, and Compression

- For example: the sentence
  The cat sat on the mat

  is encoded in ASCII as

  1010100 110100 011001 0101 .....

ASCII, stands for American Standard Code for Information Interchange.
There are 7-bit and 8-bit versions; see https://www.ascii-code.com/

- Note that the spaces are there simply to improve readability … they don't appear in the encoded version.

Text, Codes, and Compression

The following bit string is an ASCII encoded message:

1000100110010111000111101111111001001101001110111011001110
1000001101001111001101000001100101110000111100111111001

Text, Codes, and Compression

And we can decode it by chopping it into smaller strings each of 7 bits in length and by replacing the bit strings with their corresponding characters:

1000100(D)1100101(e)1100011(c)1101111(o)1100100(d)1101001(i)11011
10(n)1100111(g)0100000( )1101001(i)1110011(s)0100000( )1100101(e)110
0001(a)1110011(s)1111001(y)

Text, Codes, and Compression

- Every code can be thought of in terms of
- a finite alphabet of source symbols
- a finite alphabet of code symbols
- Each code maps every finite sequence or string of source symbols into a string of code symbols

Text, Codes, and Compression

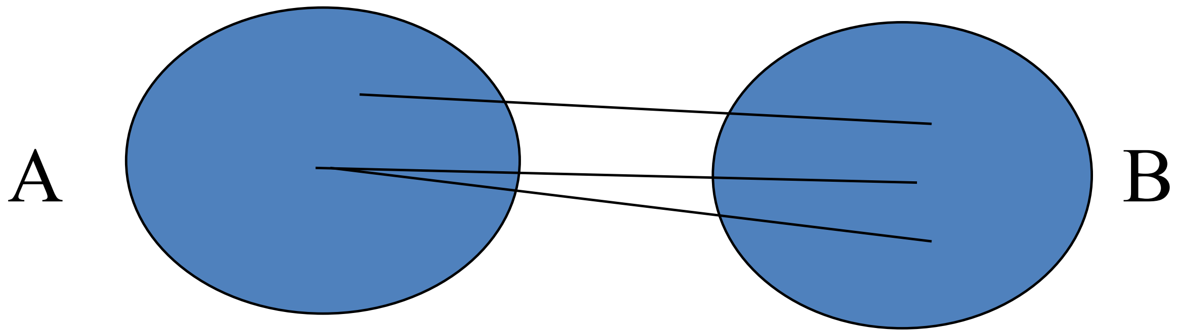
- Let A be the source alphabet
- Let B be the code alphabet
- A code f is an injective map

  $f: S_A \rightarrow S_B$

- where S_A is the set of all strings of symbols from A
- where S_B is the set of all strings of symbols from B

Text, Codes, and Compression

Injectivity ensures that each encoded string can be decoded uniquely (we do not want two source strings that are encoded as the same string)



Injective Mapping: each element in the range is related to at most one element in the domain

Text, Codes, and Compression

We are primarily interested in the code alphabet {0, 1} since we want to code source symbols strings as bit strings

Text, Codes, and Compression

- There is a problem with block codes:
  n symbols produce nb bits with a block code of length b

- For example,
  - if n = 100,000 (the number of characters in a typical 200-page book)
  - b = 7 (e.g., 7-bit ASCII code)
  - then the characters are encoded as 700,000 bits

Text, Codes, and Compression

- While we cannot encode the ASCII characters with fewer than 7 bits
- We can encode the characters with a different number of bits, depending on their frequency of occurrence
- Use fewer bits for the more frequent characters
- Use more bits for the less frequent characters
- Such a code is called a variable-length code

Text, Codes, and Compression

- First problem with variable length codes:
  - when scanning an encoded text from left to right (decoding it) …
  - How do we know when one codeword finishes and another starts?
- We require each codeword not be a prefix of any other codeword
- So, for the binary code alphabet, we should base the codes on binary code trees

Text, Codes, and Compression

- Binary code trees:

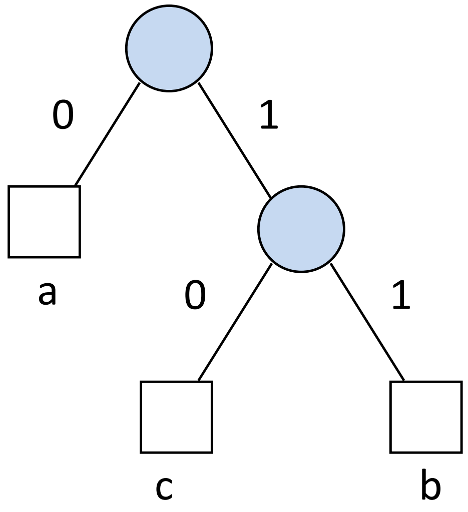
- Binary tree whose external nodes are labelled uniquely with the source alphabet symbols

- Left branches are labelled 0

- Right branches are labelled 1

Text, Codes, and Compression

A binary code tree and its prefix code



a 0
b 11
c 10

Text, Codes, and Compression

- The codeword corresponding to a symbol is the bit string given by the path from the root to the external node labeled with the symbol

- Note that, as required, no codeword is a prefix for any other codeword

  – This follows directly from the fact that source symbols are only on external nodes

  – and there is only one (unique) path to that symbol

Text, Codes, and Compression

- Codes that satisfy the prefix property are called prefix codes
- Prefix codes are important because
  - we can uniquely decode an encoded text with a left-to-right scan of the encoded text
  - by considering only the current bit in the encoded text
  - decoder uses the code tree for this purpose

Text, Codes, and Compression

- Read the encoded message bit by bit
- Start at the root
- if the bit is a 0, move left
- if the bit is a 1, move right
- if the node is external, output the corresponding symbol and begin again at the root

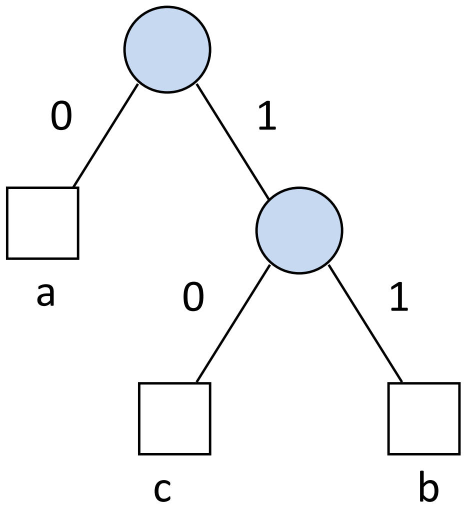
Text, Codes, and Compression

- Encoded message:

  0 0 1 1 1 0 0

- Decoded message:

  A A B C A



Optimal Variable-Length Codes

- What makes a good variable length code?

- Let $A = a_1, ..., a_n$, $n>=1$, be the alphabet of source symbols

- Let $P = p_1, ..., p_n$, $n>=1$, be their probability of occurrence

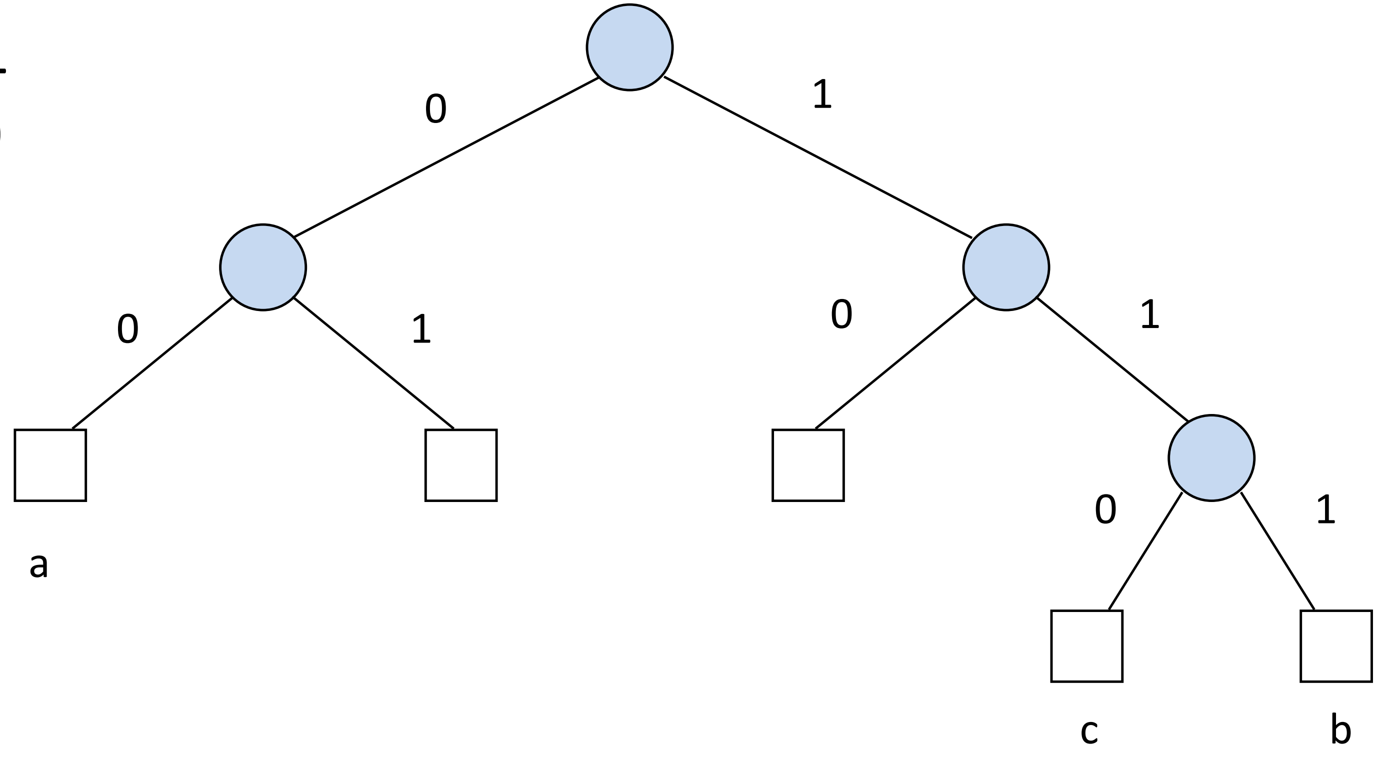
- We obtain these probabilities by analysing are representative sample of the type of text we wish to encode

Optimal Variable-Length Codes

- Any binary tree with n external nodes labelled with the n symbols defines a prefix code
- Any prefix code for the n symbols defines a binary tree with at least n external nodes
- Such a binary tree with exactly n external nodes is a reduced prefix code (tree)
- Good prefix codes are always reduced (and we can always transform a non-reduced prefix code into a reduced one)

Non-Reduced Prefix Code (Tree)

a 00
b 111
c 110



Optimal Variable-Length Codes

- Comparison of prefix codes - compare the number of bits in the encoded text
- Let $A = a_1, \ldots, a_n$, $n >= 1$, be the alphabet of source symbols
- Let $P = p_1, \ldots, p_n$ be their probability of occurrence
- Let $W = w_1, \ldots, w_n$ be a prefix code for $A = a_1, \ldots, a_n$
- Let $L = l_1, \ldots, l_n$ be the lengths of $W = w_1, \ldots, w_n$

Optimal Variable-Length Codes

- Given a source text T with $f_1, \ldots, f_n$ occurrences of $a_1, \ldots, a_n$ respectively
- The total number of bits when T is encoded is
  $\sum_{i=1}^{n} f_i l_i$
- The total number of source symbols is
  $\sum_{i=1}^{n} f_i$
- The average length of the W-encoding is

$$\text{Alength}(T, W) = \sum_{i=1}^{n} f_i l_i \Big/ \sum_{i=1}^{n} f_i$$

Optimal Variable-Length Codes

- For long enough texts, the probability p_i of a given symbol occurring isapproximately

$$p_i = f_i \Big/ \sum_{i=1}^{n} f_i$$

- So, the expected length of the W-encoding is

$$\text{Elength}(W, P) = \sum_{i=1}^{n} p_i l_i$$

Optimal Variable-Length Codes

- To compare two different codes W_1 and W_2 we can compare either
  - Alength(T, W_1) and Alength(T, W_2) or
  - Elength(W_1, P) and Elength(W_2, P)
- We say W_1 is no worse than W_2 if

  Elength(W_1, P) <= Elength(W_2, P)

- We say W_1 is optimal if

  Elength(W_1, P) <= Elength(W_2, P)
  for all possible prefix codes W_2 of A

Optimal Variable-Length Codes

- Huffman's Algorithm
- We wish to solve the following problem:
- Given n symbols $A = a_1, ..., a_n, n>=1$

  and the probability of their occurrence
  $P = p_1, ..., p_n$, respectively,

  construct an optimal prefix code for A and P

Optimal Variable-Length Codes

- This problem is an example of a global optimization problem

- Brute force (or exhaustive search) techniques are too expensive to compute:

  - Given A and P
  - Compute the set of all reduced prefix codes
  - Choose the minimal expected length prefix code

Optimal Variable-Length Codes

- This algorithm takes $O(n^n)$ time, where n is the size of the alphabet
- Why? because any binary tree of size n-1 (i.e. with n external nodes) is a valid reduced prefix tree and there are $n!$ ways of labelling the external nodes
- Since $n!$ is approximately n^n we see that there are approximately $O(n^n)$ steps to go through when constructing all the trees to check

Optimal Variable-Length Codes

- Huffman's Algorithm is only $O(n^2)$
- This is significant: if n = 128 (number of symbols in a 7-bit ASCII code)
  - $O(n^n) = 128^{128} = 5.28 \times 10^{269}$
  - $O(n^2) = 128^2 = 1.6384 \times 10^4$
  - There are 31536000 seconds in a year and if we could compute 1000 000 000 steps a second then the brute force technique would still take 1.67×10^{253} years

Optimal Variable-Length Codes

- The age of the universe is estimated to be 13 billion years, i.e., 1.3×10^{10} years
- A long way off 1.67×10^{253} years!

Optimal Variable-Length Codes

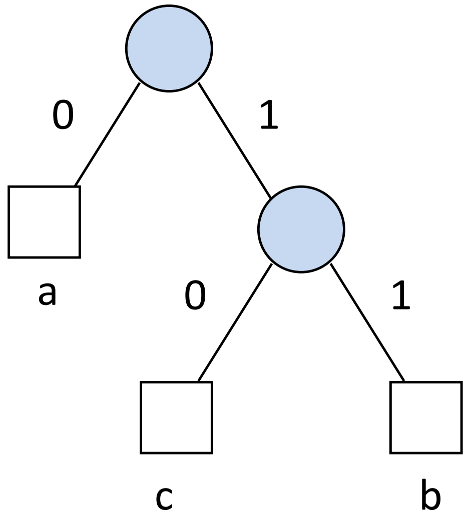
- Huffman's Algorithm uses a technique called Greedy
- It uses local optimization to achieve a globally optimum solution
  - Build the code incrementally
  - Reduce the code by one symbol at each step
  - Merge the two symbols that have the smallest probabilities into one new symbol

Optimal Variable-Length Codes

- Before we begin, note that we'd like a tree with the symbols which have the lowest probability to be on the longest path
- Why?
- Because the length of the codeword is equal to the path length and we want
  - short codewords for high-probability symbols
  - longer codewords for low-probability symbols

Text, Codes, and Compression

A binary code tree and its prefix code



a 0
b 11
c 10

Huffman's Algorithm

- We will treat Huffman's Algorithm for just six letters, i.e, n = 6, and there are six symbols in the source alphabet

- These are, with their probabilities,

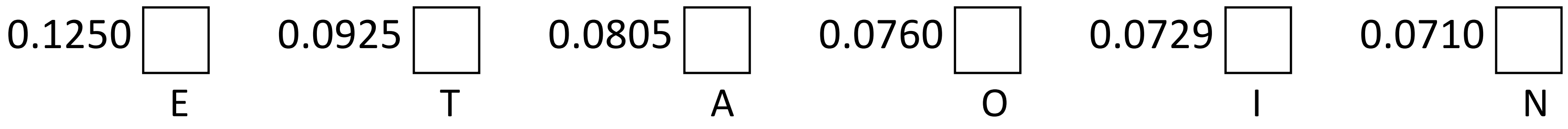
E	0.1250
T	0.0925
A	0.0805
O	0.0760
I	0.0729
N	0.0710

Huffman’s Algorithm

Step 1:

- Create a forest of code trees, one for each symbol

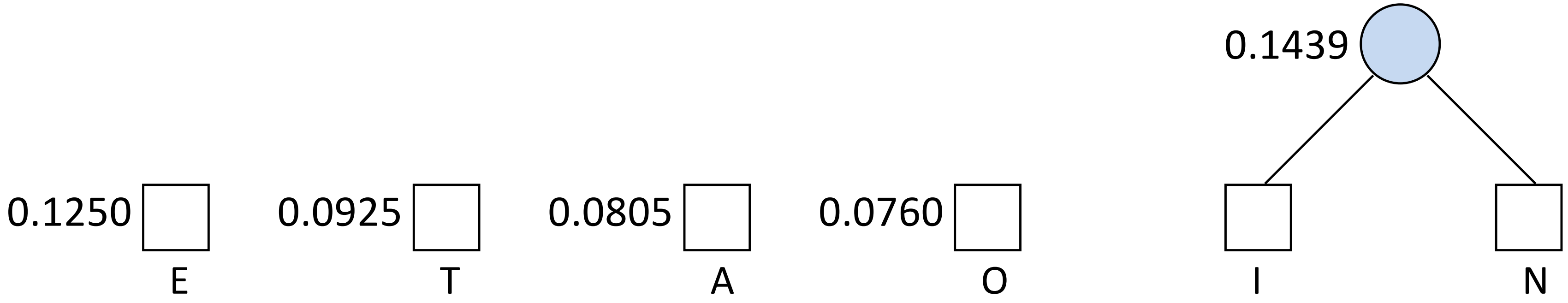
- Each tree comprises a single external node (empty tree) labelled with its symbol and weight (probability)



Huffman's Algorithm

Step 2:

- Choose the two binary trees, B1 and B2, that have the smallest weights
- Create a new root node with B1 and B2 as its children and with weight equal to the sum of these two weights

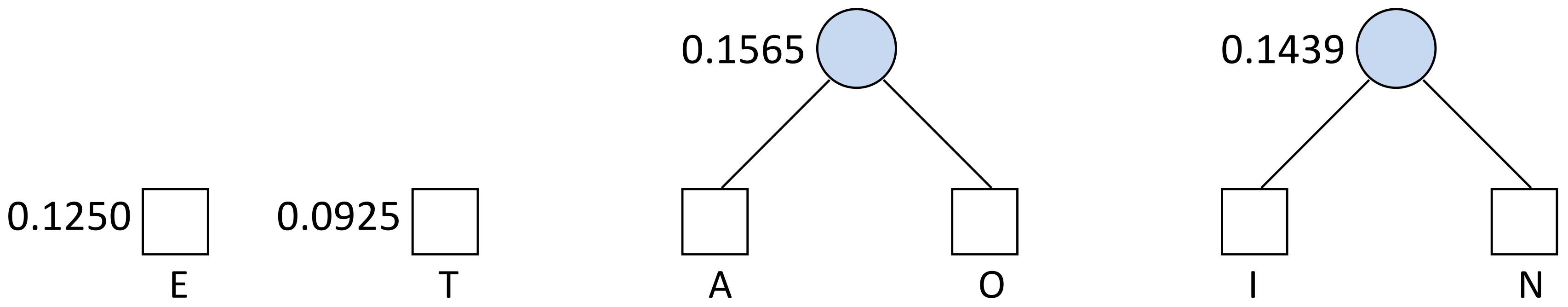


Huffman’s Algorithm

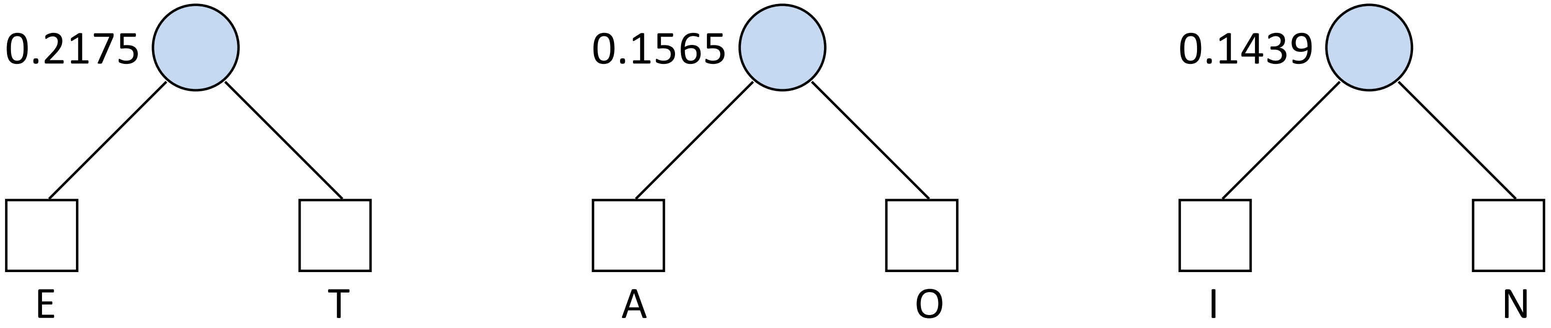
Step 3:

– Repeat step 2!

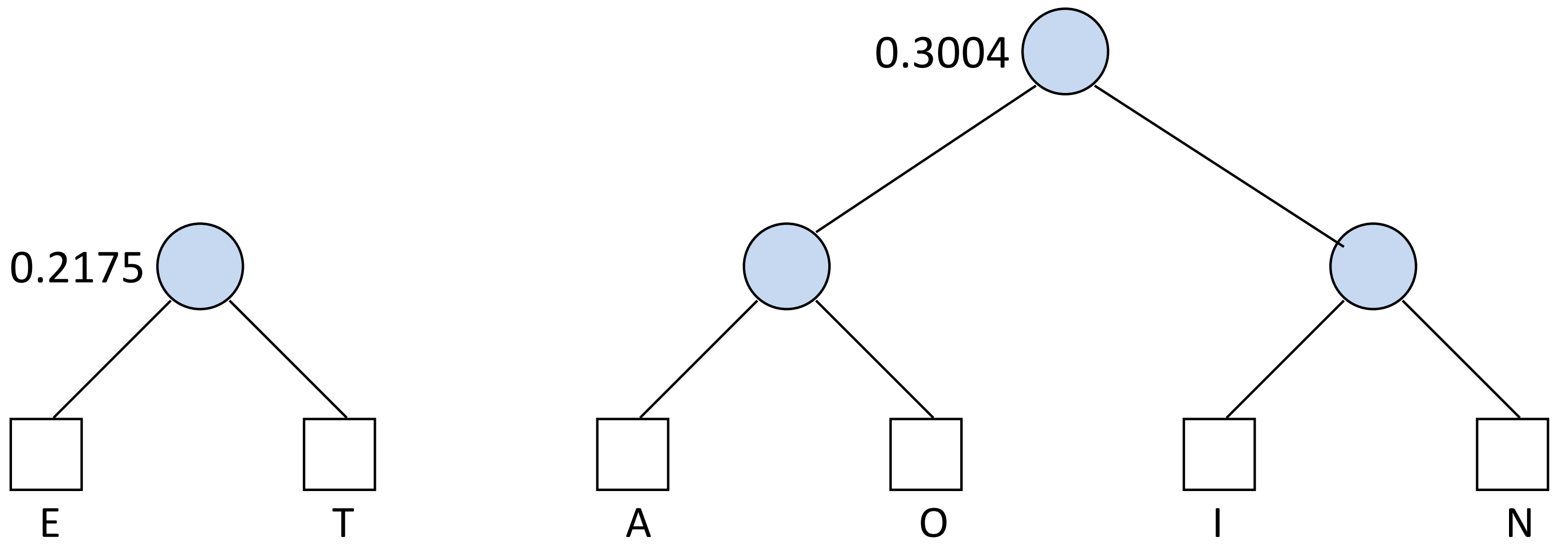
Huffman's Algorithm



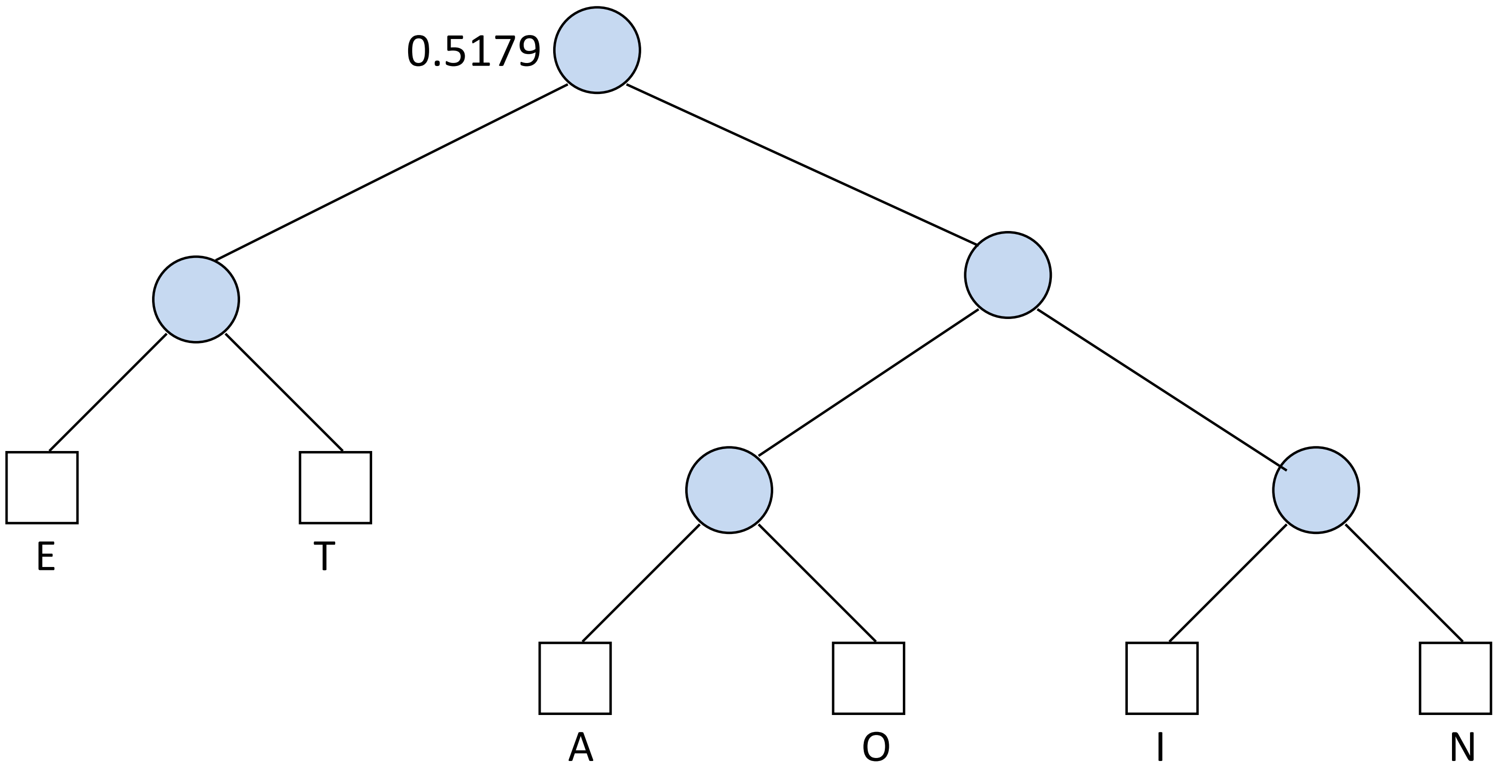
Huffman's Algorithm



Huffman's Algorithm



Huffman's Algorithm



Huffman's Algorithm

The final prefix code is:

A	100
E	00
I	110
N	111
O	101
T	01

Huffman's Algorithm

Three phases in the algorithm

1. Initialize the forest of code trees
2. Construct an optimal code tree
3. Compute the encoding map

Huffman's Algorithm

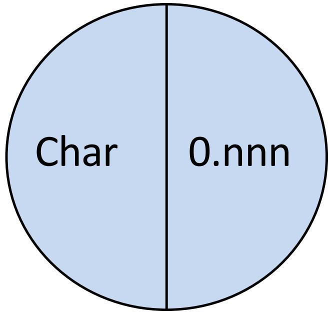
Phase 1: Initialize the forest of code trees

- How will we represent the forest of trees?
- Better question: how will we represent our tree … have to store both alphanumeric characters and probabilities?
- Need some kind of composite node
- Opt to represent this composite node as an INTERNAL node

Huffman's Algorithm

– Consequently, the initial tree is simply one internal node

– That is, it is a root (with two external nodes)



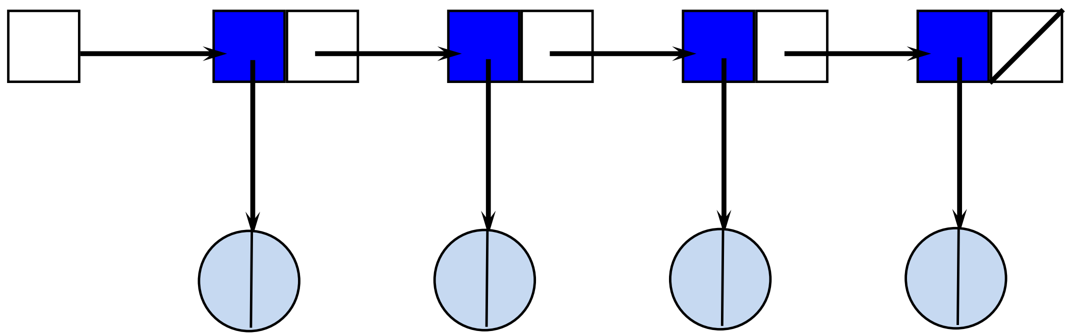
Huffman's Algorithm

So, to create such a tree we simply invoke the following operations:

- Initialize the tree … tree()
- Add a node … addnode(char, weight, T)

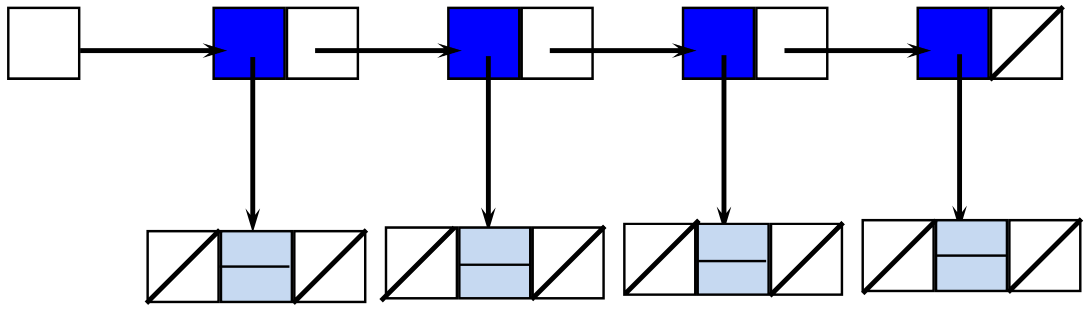
Huffman's Algorithm

- We must also keep track of our forest
- We could represent it as a linked list of pointers to Binary trees …



Huffman's Algorithm

Represented as:



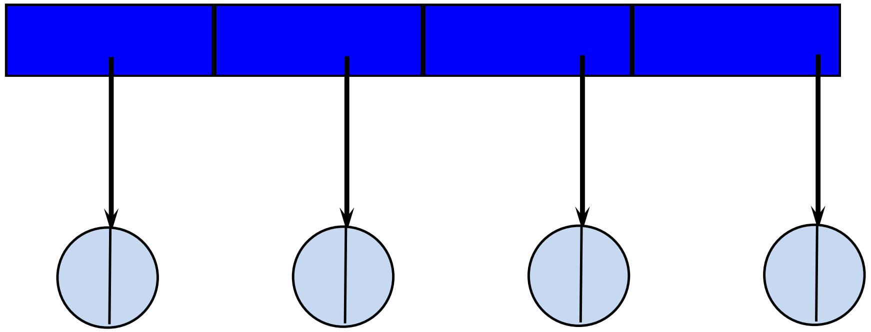
Huffman's Algorithm

- Is there an alternative?
- Question: why do we use dynamic data structures?
- Answer:
  - When we don't know in advance how many elements are in our data set
  - When the number of elements varies significantly
- Is this the case here?
- No!

Huffman's Algorithm

- So, our alternatives are? ……
- An array, indexed by number, of type …
- binary_tree, i.e., each element in the array can point to a binary code tree

Huffman's Algorithm



Huffman's Algorithm

- What will be the dimension of this array?

- n, the number of symbols in our source alphabet since this is the number of trees we start out with in our forest initially

Huffman's Algorithm

Phase 2: construct the optimal code tree

Huffman's Algorithm

```
Find the tree with the smallest weight - A, at element i
Find the tree with the next smallest weight - B, at element j

Construct a tree, with right sub-tree A, left sub-tree B,
  with root having weight = sum of the roots of A and B

Let array element i point to the new tree
Remove tree at element j
(delete it if you made a copy of left sub-tree B)
```

Huffman's Algorithm

let n be the number of trees initially

Repeat

```
    Find the tree with the smallest weight - A, at element i
    Find the tree with the next smallest weight - B, at element j

    Construct a tree, with right sub-tree A, left sub-tree B,
      with root having weight = sum of the roots of A and B

    Let array element i point to the new tree
    Remove tree at element j
    (delete it if you made a copy of left sub-tree B)
```

Until only one tree left in the array

Huffman's Algorithm

Phase 3: Compute the encoding map

- We need to write out a list of source symbols together with their prefix code
- We need to write out the contents of each external node (or each frontier internal node) together with the path to that node
- We need to **traverse** the binary code tree in some manner

But .... we want to print out the symbol and the prefix code:

i.e., the symbol at the leaf node

and the path by which we got to that node

- How will we represent the path?
- As an array of binary values
(representing the left and right links on the path)