Data Structures and Algorithms for Engineers

Module 7: Graphs

Lecture 5: Shortest Path Algorithms, Dijkstra's algorithm, Floyd's algorithm

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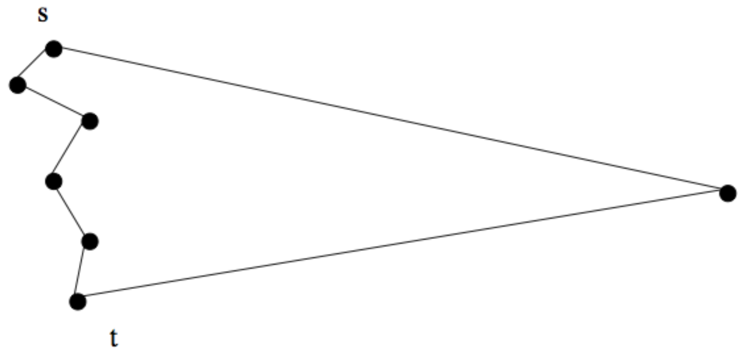
Shortest Paths

- A path is a sequence of edges connecting two vertices
- The shortest path in an unweighted graph can be constructed using Breadth-First Search
- Also works if the weights are all the same in a weighted graph
- Does not work for weighted graphs in general:
  the shortest route (e.g., in time) may pass through many intermediate vertices



Shortest Paths

Two main algorithms for finding shortest paths

- Dijkstra's Algorithm
  - Finds shortest path between start and destination vertices
    - Some implementations find the shortest path between a start vertex and all other vertices, i.e., shortest path spanning tree rooted in the start vertex
  - $O(n^2)$ for with simple data structures
- Floyd's Algorithm
  - Finds shortest path between all pairs of vertices in a graphs
  - $O(n^3)$

Shortest Paths

Dijkstra’s Algorithm

- Greedy algorithm
- Repeatedly select the smallest weight edge that will extend the path
  - Start from given vertex,
  - Extend the path, one edge at a time,
  - Until all vertices are included
- Very similar to Prim’s algorithm, except …
  - Instead of just considering the weight of the next potential edge
  - We also have to consider **the distance from the start to the vertex from which that edge emanates**

Shortest Paths

```
ShortestPath-Dijkstra(G, s, t)

path= {s}
for i = 1 to n, dist[i] = ∞
for each edge (s, v), dist[v] = w(s, v)  // initial distances are just the weights
last = s
while (last != t)
    select $v_{next}$, the unknown vertex minimizing dist[v]
    for each edge ($v_{next}$, x), dist[x] = min[dist[x], dist[$v_{next}$] + w($v_{next}$, x)]
    last = $v_{next}$
    path = path ∪ {$v_{next}$}
```

Shortest Paths

```
ShortestPath-Dijkstra(G, s, t)

path= {s}
for i = 1 to n, dist[i] = ∞
for each edge (s, v), dist[v] = w(s, v)  // initial distances are just the weights
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    last = $v_{next}$
    path = path ∪ {$v_{next}$}
```

In Prim's algorithm,
they were always the weights

but in Dijkstra they are the
shortest distance to that vertex (so far)

Shortest Paths

```
ShortestPath-Dijkstra(G, s, t)

path= {s}
for i = 1 to n, dist[i] = ∞
for each edge (s, v), dist[v] = w(s, v)  // initial distances are just the weights
last = s
while (last != t)
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```

Extend the path from the vertex with the shortest distance so far

Shortest Paths

```
ShortestPath-Dijkstra(G, s, t)

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for i = 1 to n, dist[i] = $\infty$
for each edge (s, v), dist[v] = w(s, v)  // initial distances are just the weights
last = s
while (last != t)
    select $v_{next}$, the unknown vertex minimizing dist[v]
    for each edge ($v_{next}$, x), dist[x] = min[dist[x], dist[$v_{next}$] + w($v_{next}$, x)]
    last = $v_{next}$
    path = path ∪ {$v_{next}$}
```

We now have a new way of reaching x ...

... so update the (total) distance to x ...

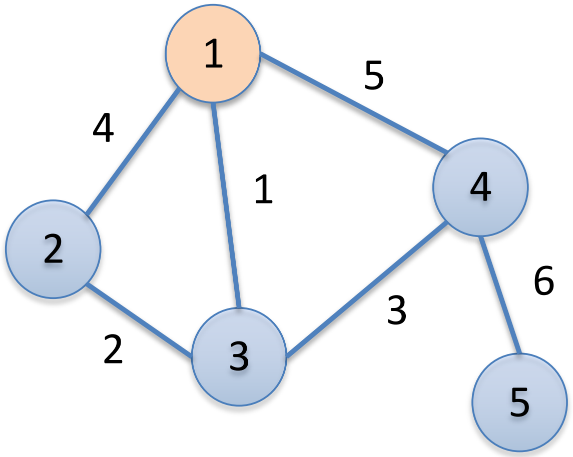
... but only if it is less than the current distance

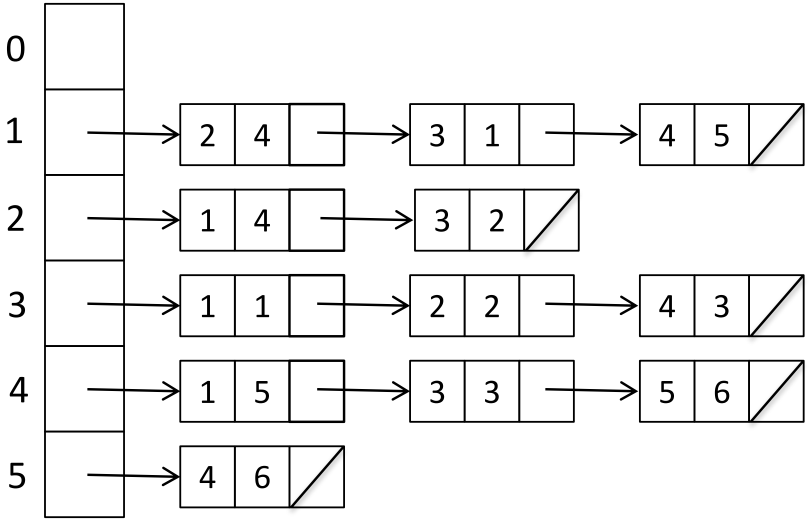
Shortest Paths

```
/* Dijkstra’s algorithm: implementation based of Prim’s algorithm */

dijkstra(graph *g, int start) {
   int i;                 /* counter                          */
   edgenode *p;           /* temporary pointer                */
   bool intree[MAXV+1];   /* is the vertex in the tree yet?   */
   int distance[MAXV+1];  /* cost of adding to tree           */
   int parent[MAXV+1];    /* parent vertex                    */
   int v;                 /* current vertex to process        */
   int w;                 /* candidate next vertex            */
   int weight;            /* edge weight                      */
   int dist;              /* best current distance from start */

   for (i=1; i<=g->nvertices; i++) {
      intree[i] = FALSE;
      distance[i] = MAXINT;
      parent[i] = -1;
   }

   distance[start] = 0;
   v = start;
```

Shortest Paths

```
   while (intree[v] == FALSE) {
      intree[v] = TRUE;
      p = g->edges[v];
      while (p != NULL) {
         w = p->y;
         weight = p->weight;
         if ((distance[v] + weight < distance[w])) { //changes from Prim
            distance[w] = distance[v] + weight;
            parent[w] = v;
         }
         p = p->next;
      }
      v = 1;
      dist = MAXINT;
      for (i=1; i<=g->nvertices; i++)
         if ((intree[i] == FALSE) && (distance[i] < dist)) {
            dist = distance[i];
            v = i;
         }
   }
}
```

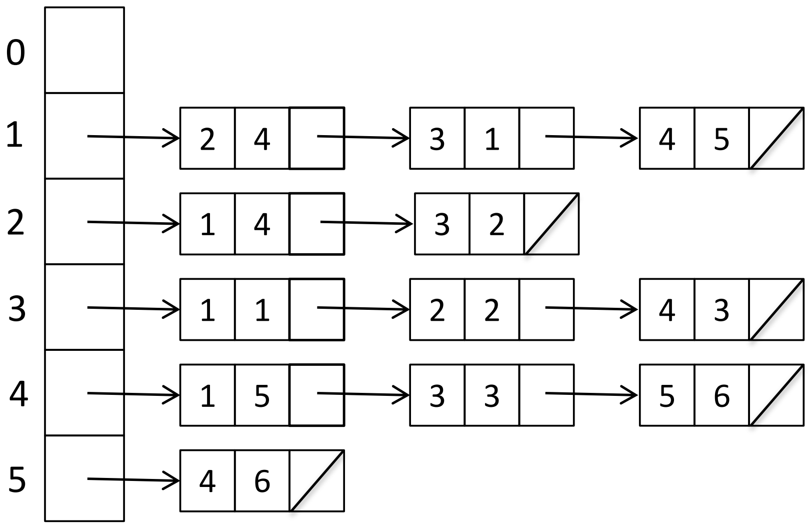




0			
1	2 4	3 1	4 5
2	1 4	3 2	
3	1 1	2 2	4 3
4	1 5	3 3	5 6
5	4 6		

	intree	distance	parent	v	w	weight	dist
0							
1							
2							
3							
4							
5							

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```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1 → 2 4 → 3 1 → 4 5
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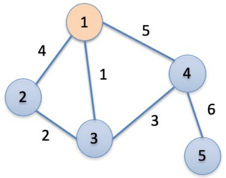
3 → 1 1 → 2 2 → 4 3
4 → 1 5 → 3 3 → 5 6
5 → 4 6



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0							
1							
2							
3							
4							
5							

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      w = p->y;
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      }
}
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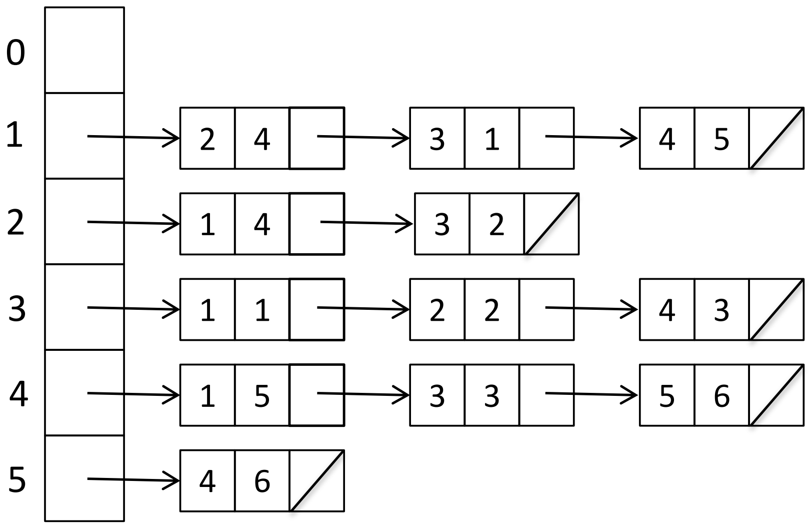

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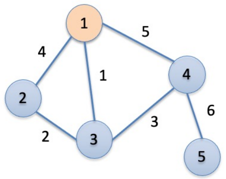

	intree	distance	parent	v	w	weight	dist
0							
1	F	∞	-1				
2	F	∞	-1				
3	F	∞	-1				
4	F	∞	-1				
5	F	∞	-1				

```
for (i=1; i<=g->nvertices; i++) {
    intree[i] = FALSE;
    distance[i] = MAXINT;
    parent[i] = -1;
}

distance[start] = 0;
v = start;
                   start = 1

while (intree[v] == FALSE) {

    intree[v] = TRUE;
    p = g->edges[v];

    while (p != NULL) {
        w = p->y;
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        if ((distance[v]+weight < distance[w])) {
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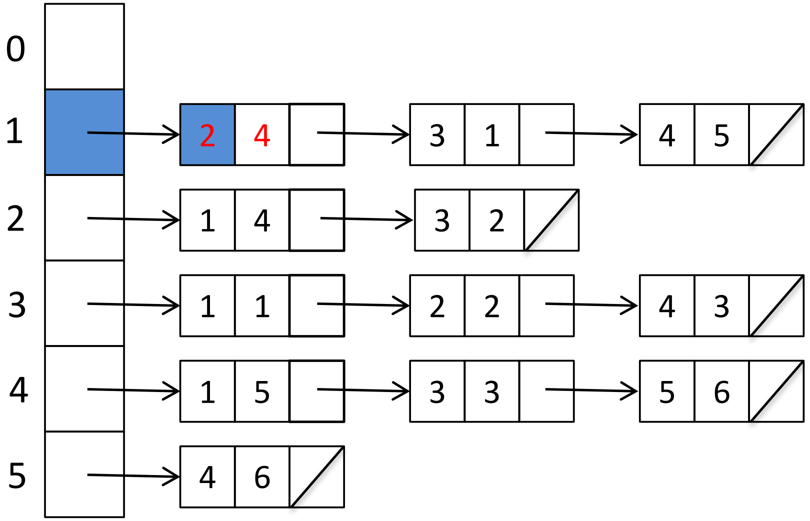

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4 → 1 5 → 3 3 → 5 6

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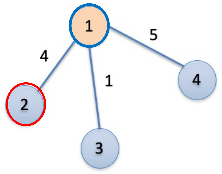

	intree	distance	parent
0			
1	T	0	-1
2	F	∞	-1
3	F	∞	-1
4	F	∞	-1
5	F	∞	-1

v	w	weight	dist
1	2	4	

```
for (i=1; i<=g->nvertices; i++) {
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distance[start] = 0;
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   }

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}
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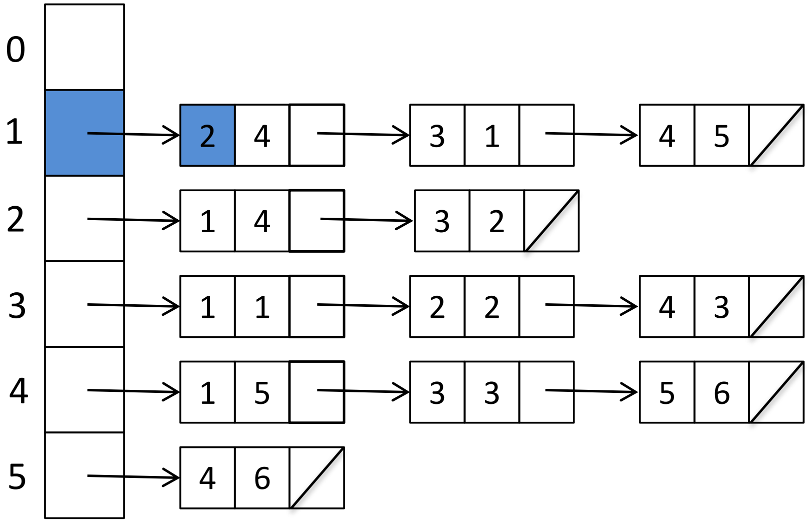

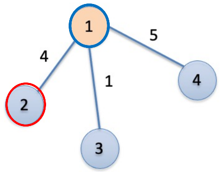

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0			
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3	F	∞	-1
4	F	∞	-1
5	F	∞	-1

v	w	weight	dist
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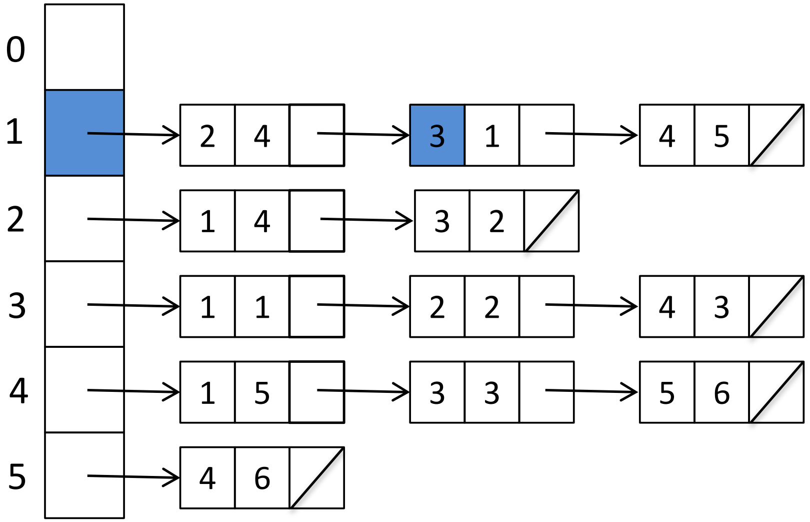

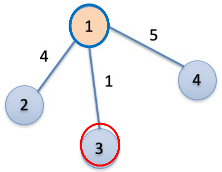

	intree	distance	parent
0			
1	T	0	-1
2	F	4	1
3	F	1	1
4	F	∞	-1
5	F	∞	-1

v	w	weight	dist
1	2	4	

```
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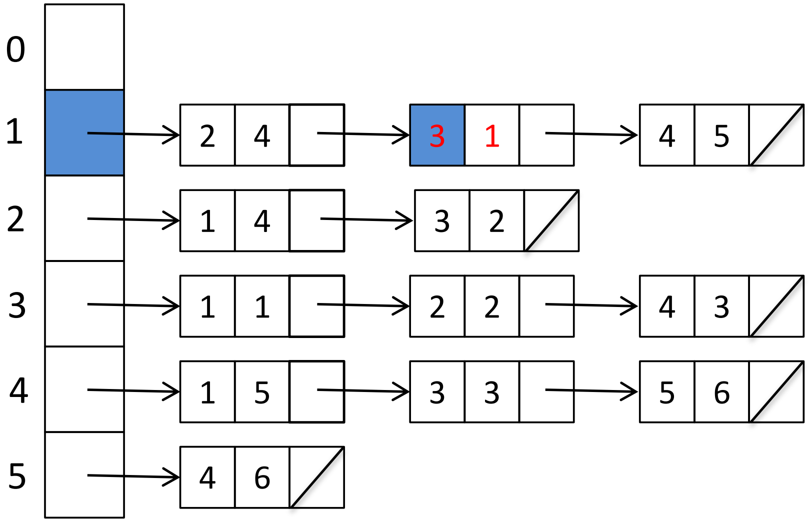

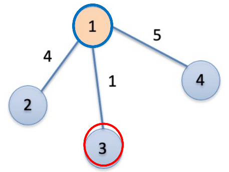

	intree	distance	parent
0			
1	T	0	-1
2	F	4	1
3	F	1	1
4	F	∞	-1
5	F	∞	-1

v	w	weight	dist
1	2	4	
	3	1	

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for (i=1; i<=g->nvertices; i++) {
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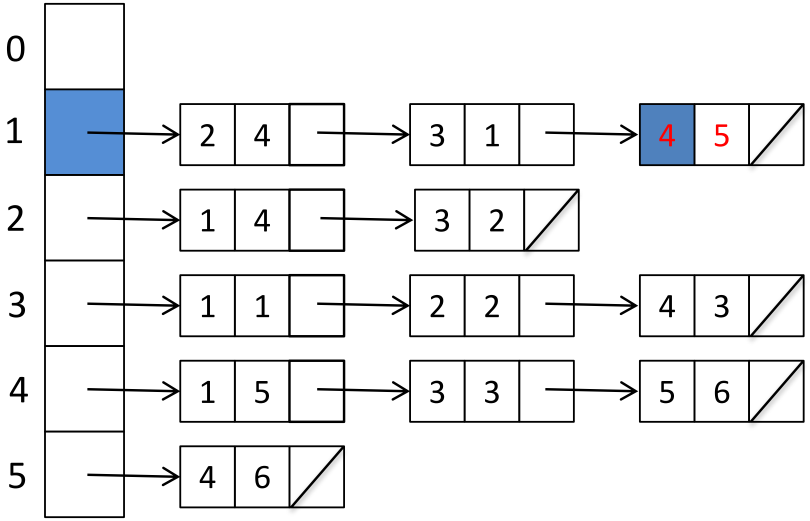

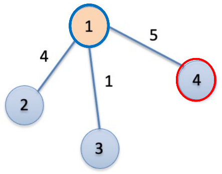

	intree	distance	parent
0			
1	T	0	-1
2	F	4	1
3	F	1	1
4	F	5	1
5	F	∞	-1

v	w	weight	dist
1	2	4	
	3	1	
	4	5	

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for (i=1; i<=g->nvertices; i++) {
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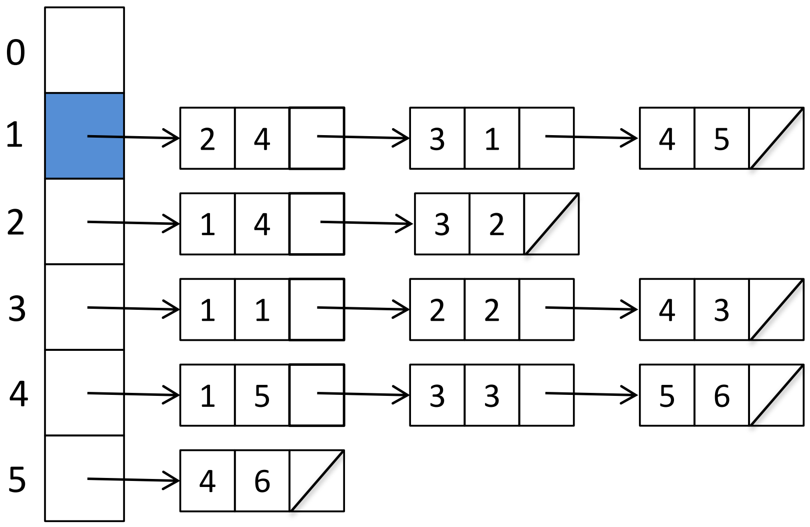

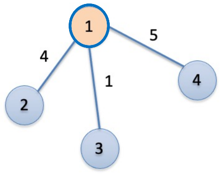

	intree	distance	parent
0			
1	T	0	-1
2	F	4	1
3	F	1	1
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v	w	weight	dist
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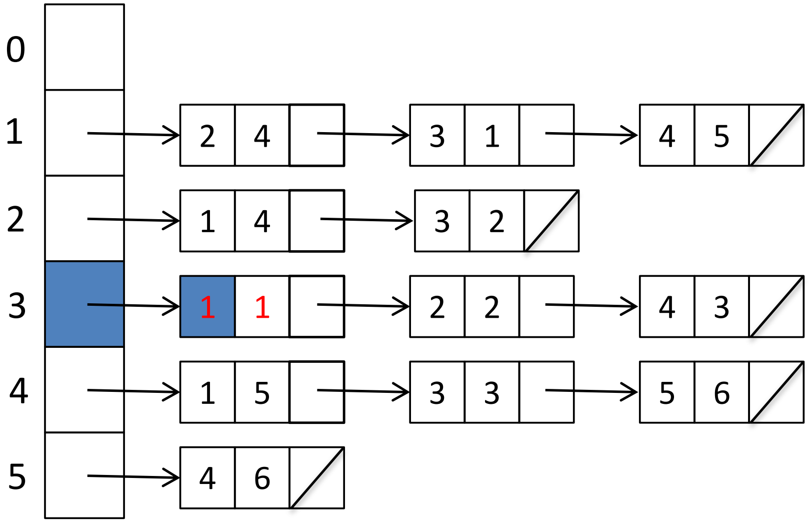

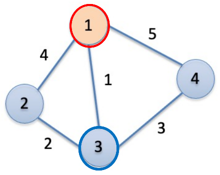

	intree	distance	parent
0			
1	T	0	-1
2	F	4	1
3	T	1	1
4	F	5	1
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	

```
for (i=1; i<=g->nvertices; i++) {
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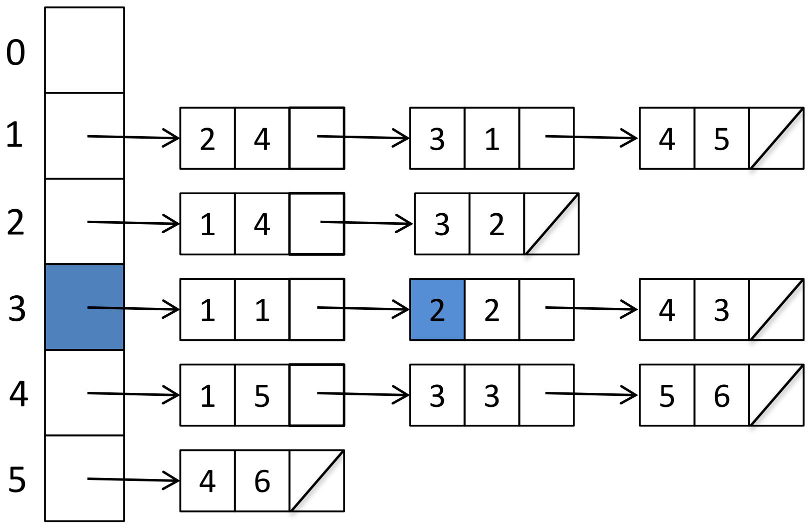

0

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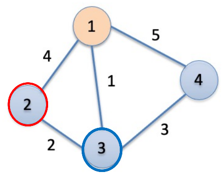

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1	T	0	-1
2	F	4	1
3	T	1	1
4	F	5	1
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	

```
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      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```
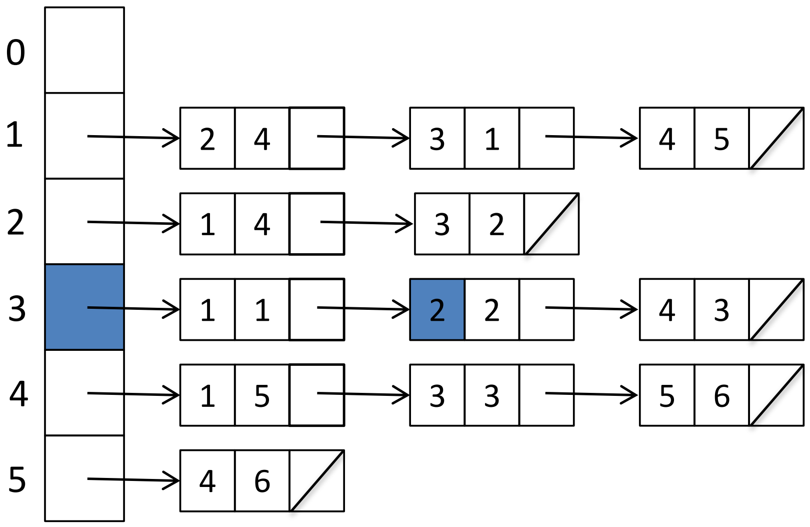
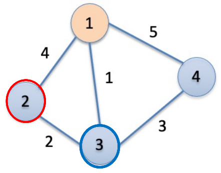
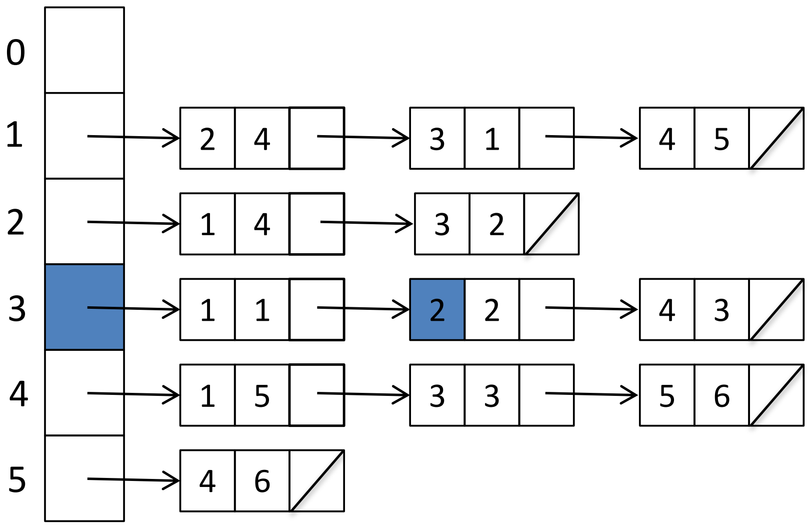


0
1 → 2 4 → 3 1 → 4 5
2 → 1 4 → 3 2
3 → 1 1 → 2 2 → 4 3
4 → 1 5 → 3 3 → 5 6
5 → 4 6

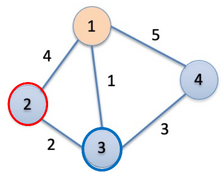


	intree	distance	parent
0			
1	T	0	-1
2	F	4	1
3	T	1	1
4	F	5	1
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	
	2	2	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```



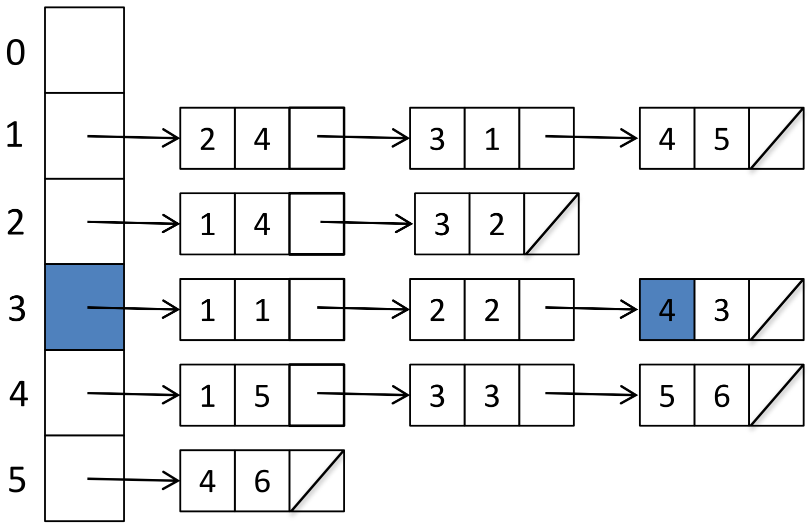


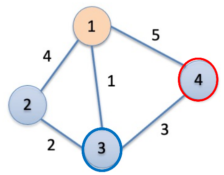
	intree	distance	parent
0			
1	T	0	-1
2	F	3	3
3	T	1	1
4	F	5	1
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	
	2	2	

We now have a shorter path to vertex 2

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])){
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

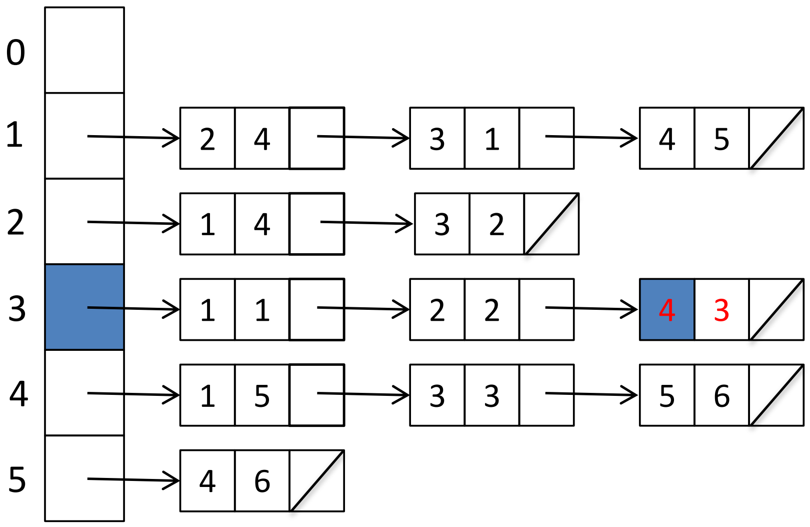


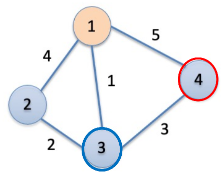


	intree	distance	parent
0			
1	T	0	-1
2	F	3	3
3	T	1	1
4	F	5	1
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	
	2	2	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

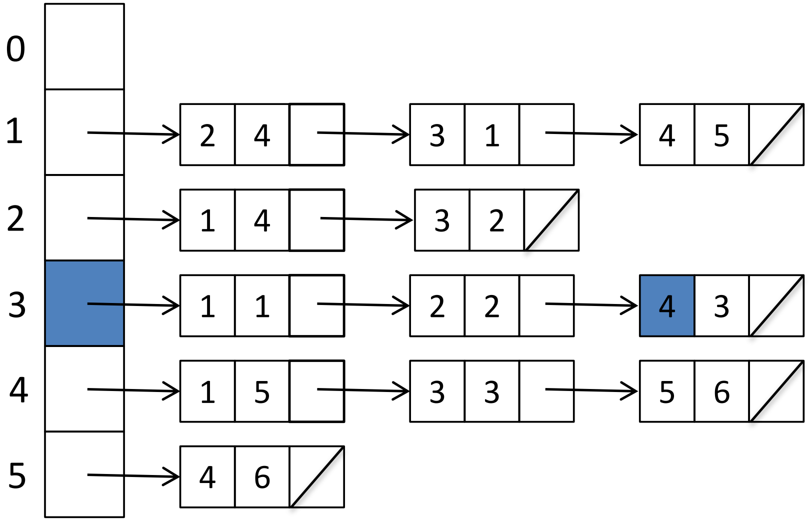


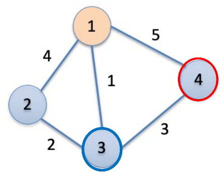


	intree	distance	parent
0			
1	T	0	-1
2	F	3	3
3	T	1	1
4	F	5	1
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	
	2	2	
	4	3	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```



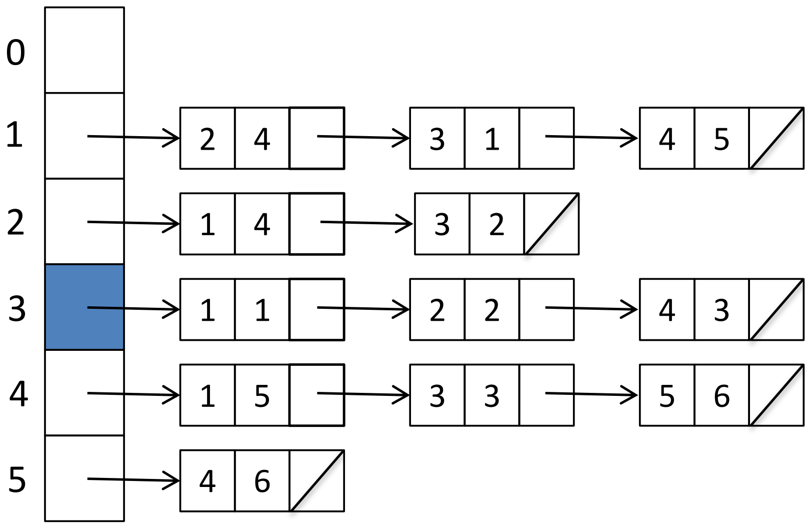


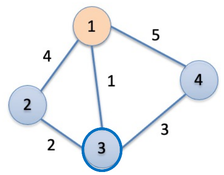
	intree	distance	parent
0			
1	T	0	-1
2	F	3	3
3	T	1	1
4	F	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	
	2	2	
	4	3	

We now have a shorter path to vertex 4

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

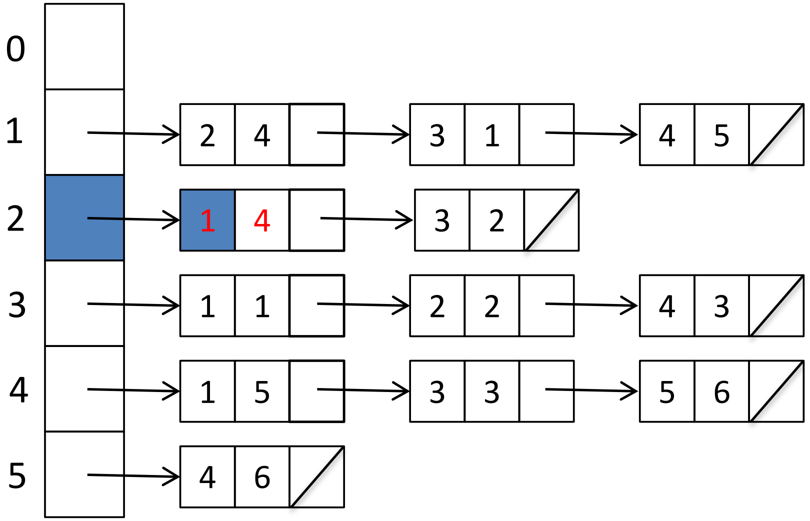


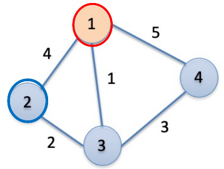


	intree	distance	parent
0			
1	T	0	-1
2	F	3	3
3	T	1	1
4	F	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

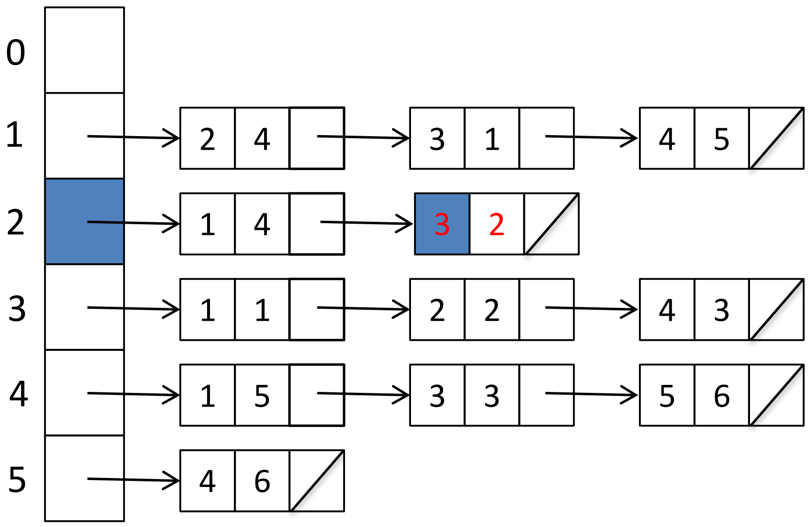


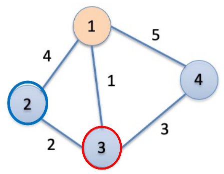


	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	F	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	
	1	4	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

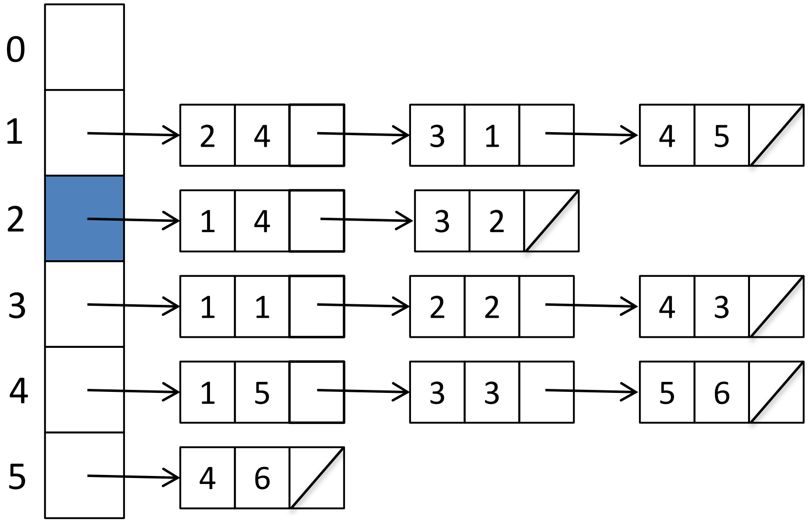


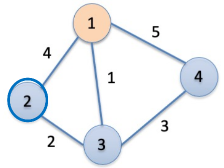


	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	F	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	
	1	4	
	3	2	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

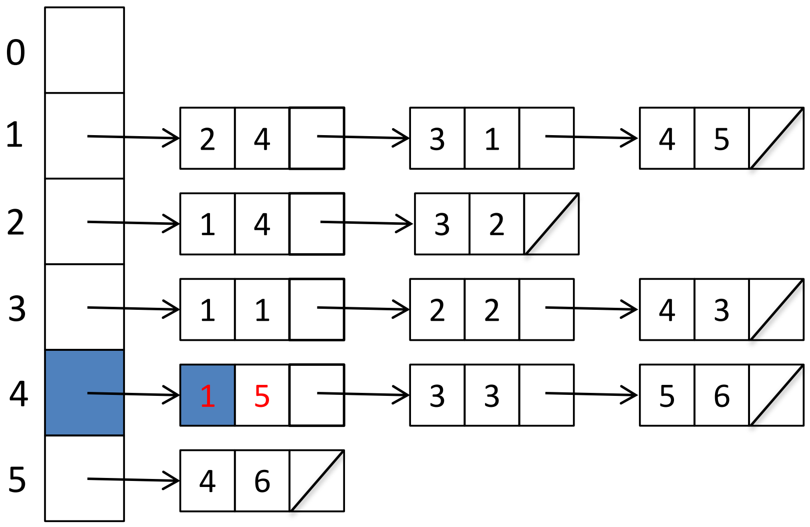


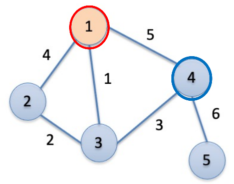


	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	F	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	∞
1	1	4	3
4	3	2	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

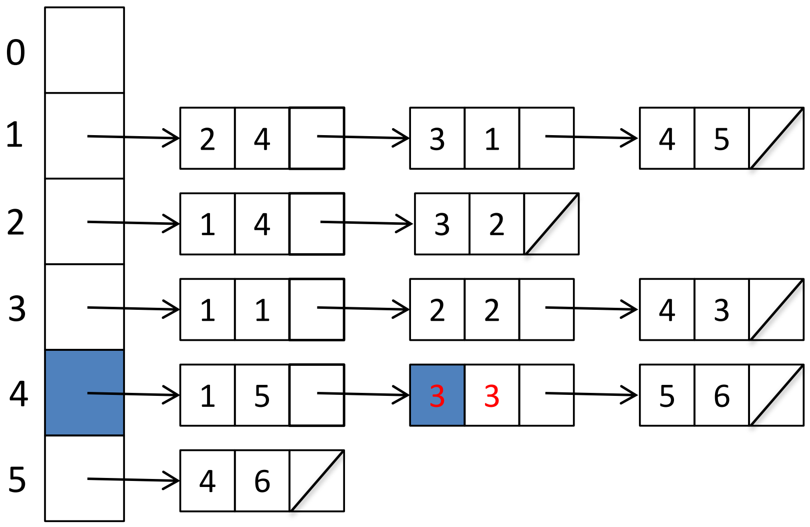


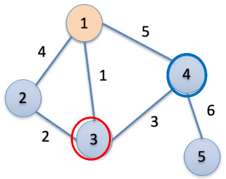


	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	T	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	∞
1	1	4	3
4	3	2	
	1	5	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```

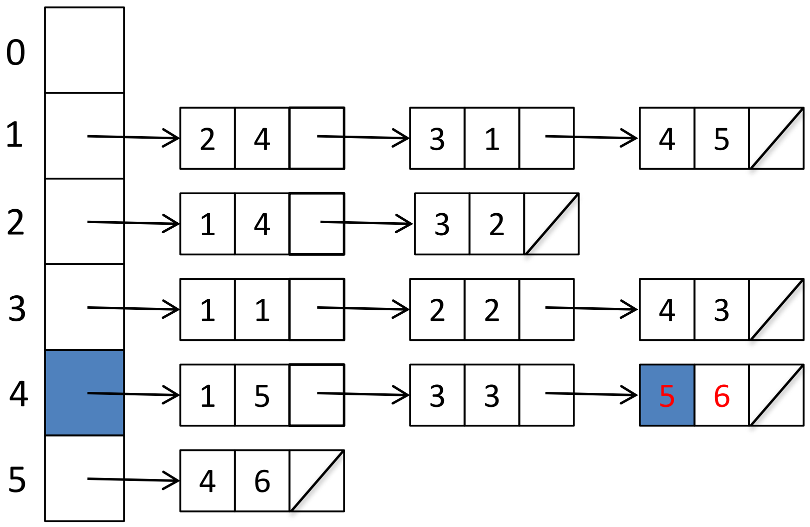


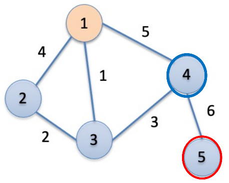


	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	T	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	∞
1	1	4	3
4	3	2	
	1	5	
	3	3	

```
for (i=1; i<=g->nvertices; i++) {
    intree[i] = FALSE;
    distance[i] = MAXINT;
    parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

    intree[v] = TRUE;
    p = g->edges[v];

    while (p != NULL) {
        w = p->y;
        weight = p->weight;
        if ((distance[v]+weight < distance[w])) {
            distance[w] = distance[v]+weight ;
            parent[w] = v;
        }
        p = p->next;
    }

    v = 1;
    dist = MAXINT;
    for (i=1; i<=g->nvertices; i++)
        if ((intree[i] == FALSE) &&
             (distance[i] < dist)) {
            dist = distance[i];
            v = i;
        }
}
```

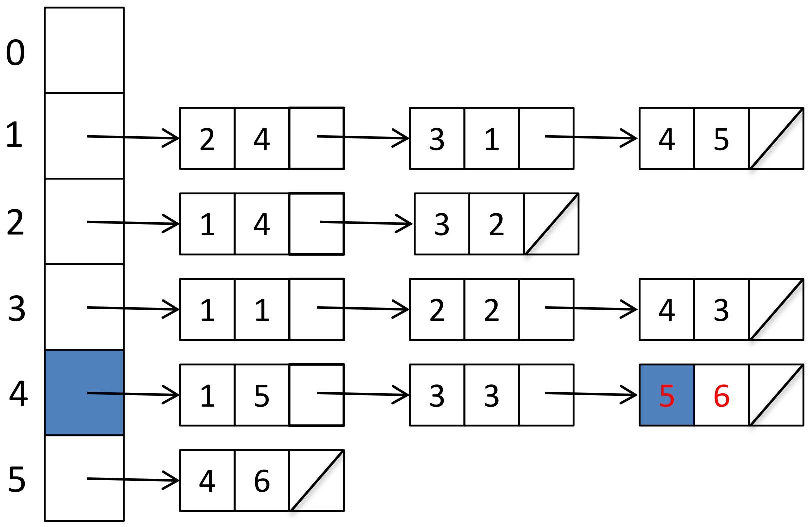


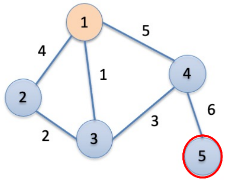


	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	T	4	3
5	F	∞	-1

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	∞
1	1	4	3
4	3	2	
	1	5	
	3	3	
	5	6	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```



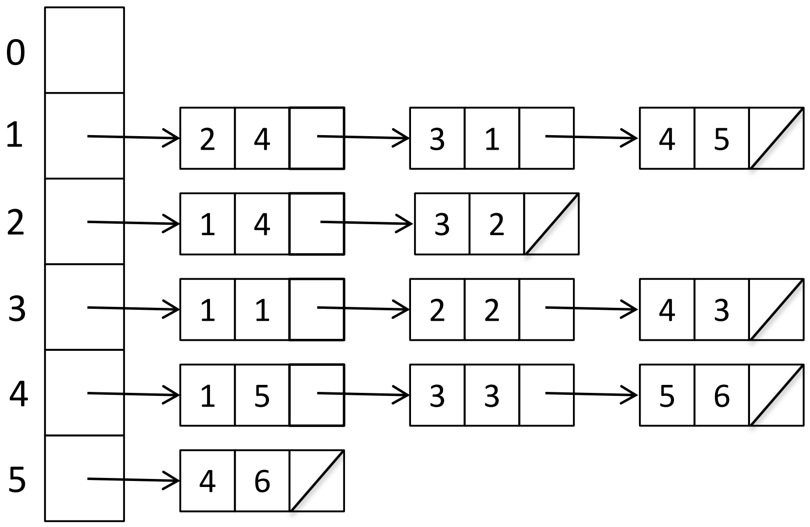


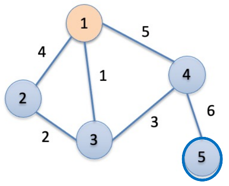
	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	T	4	3
5	F	10	4

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	∞
1	1	4	3
4	3	2	
	1	5	
	3	3	
	5	6	

We now have a shorter path to vertex 5

```
for (i=1; i<=g->nvertices; i++) {
    intree[i] = FALSE;
    distance[i] = MAXINT;
    parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

    intree[v] = TRUE;
    p = g->edges[v];

    while (p != NULL) {
        w = p->y;
        weight = p->weight;
        if ((distance[v]+weight < distance[w])) {
            distance[w] = distance[v]+weight ;
            parent[w] = v;
        }
        p = p->next;
    }

    v = 1;
    dist = MAXINT;
    for (i=1; i<=g->nvertices; i++)
        if ((intree[i] == FALSE) &&
             (distance[i] < dist)) {
            dist = distance[i];
            v = i;
        }
}
```

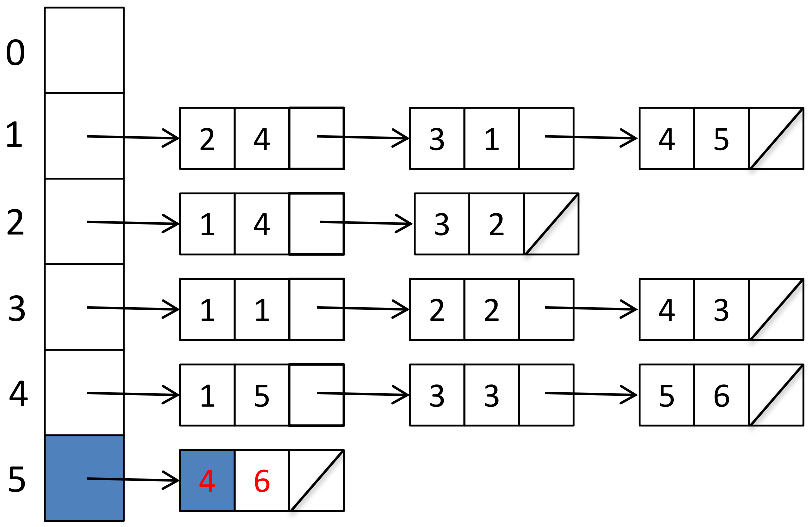




	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	T	4	3
5	F	10	4

v	w	weight	dist
1	2	4	∞
1	3	1	4
2	4	5	1
3	1	1	∞
1	2	2	2
2	4	3	∞
1	1	4	3
4	3	2	∞
1	1	5	6
5	3	3	
	5	6	

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```



0

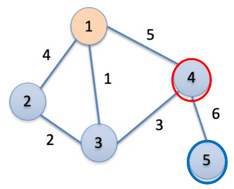
1 → 2 4 → 3 1 → 4 5

2 → 1 4 → 3 2

3 → 1 1 → 2 2 → 4 3

4 → 1 5 → 3 3 → 5 6

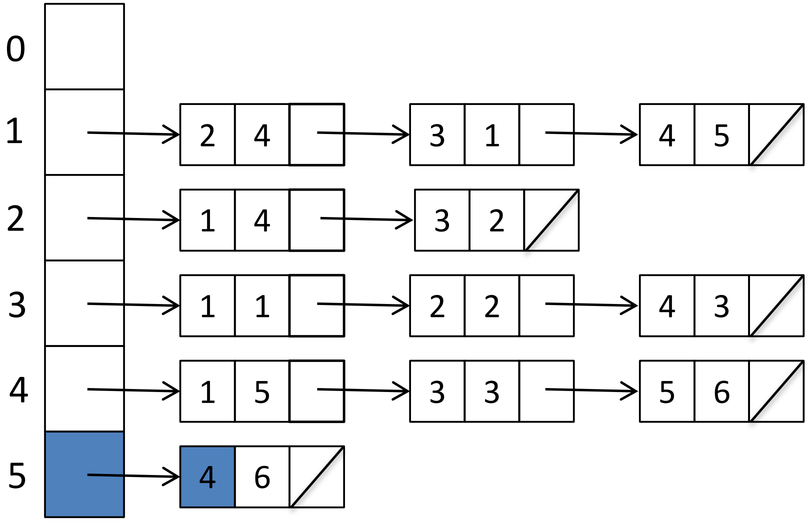
5 → 4 6

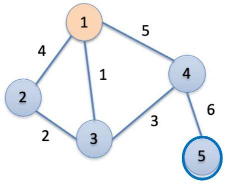


	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	T	4	3
5	T	10	4

v	w	weight	dist
1	…	…	∞
1	4	6	4
2			1
3			∞
1			2
2			∞
1			3
4			∞
1			6
5			

```
for (i=1; i<=g->nvertices; i++) {
   intree[i] = FALSE;
   distance[i] = MAXINT;
   parent[i] = -1;
}

distance[start] = 0;
v = start;

while (intree[v] == FALSE) {

   intree[v] = TRUE;
   p = g->edges[v];

   while (p != NULL) {
      w = p->y;
      weight = p->weight;
      if ((distance[v]+weight < distance[w])) {
         distance[w] = distance[v]+weight ;
         parent[w] = v;
      }
      p = p->next;
   }

   v = 1;
   dist = MAXINT;
   for (i=1; i<=g->nvertices; i++)
      if ((intree[i] == FALSE) &&
          (distance[i] < dist)) {
         dist = distance[i];
         v = i;
      }
}
```





	intree	distance	parent
0			
1	T	0	-1
2	T	3	3
3	T	1	1
4	T	4	3
5	T	10	4

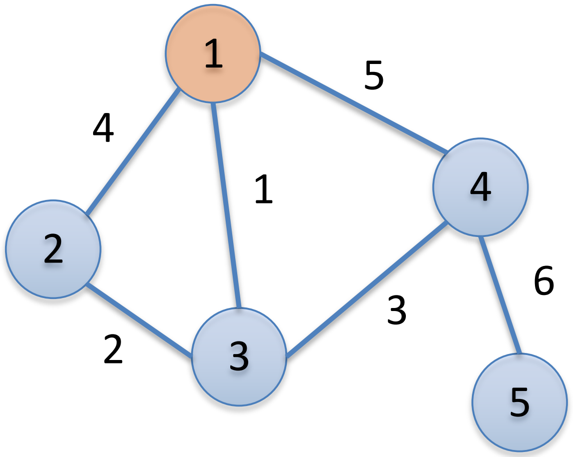
v	w	weight	dist
1	…	…	∞
1	4	6	4
2			1
3			∞
1			2
2			∞
1			3
4			∞
1			6
5			∞
1			

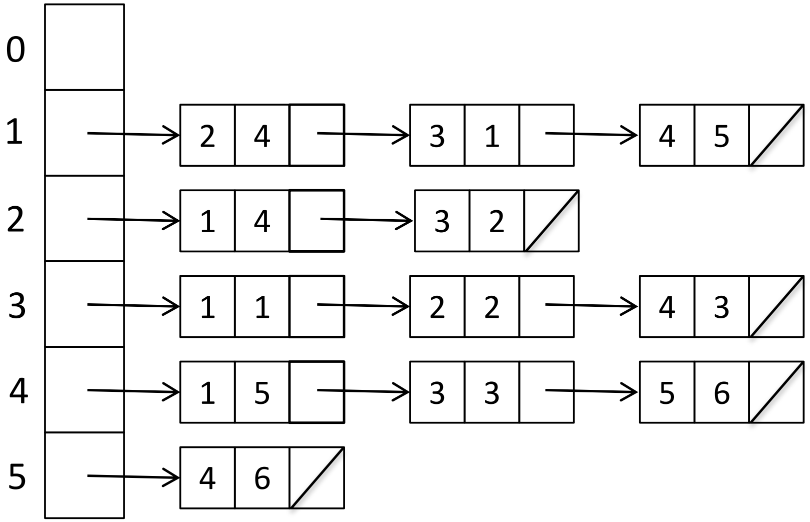
Shortest Paths

Dijkstra's Algorithm

- This implementation finds the shortest path spanning tree, i.e. shortest path between a `start` vertex and all other vertices
- The length of the shortest path from `start` to a given vertex `t` is exactly the value of `distance[t]`
- To find the actual path, follow the `parent` relations from `t` until we hit `start` (or -1 if no such path exists)
- We did this in Breadth-First Search

```
find_path(int start, int end, int parents[])
```





	parent	distance
0		
1	-1	0
2	3	3
3	1	1
4	3	4
5	4	10

Shortest Paths

All-Pairs Shortest Path – Floyd's Algorithm

- Find the centre vertex in a graph
  - Minimize **longest or average distance to all other nodes**
  - e.g. good place to set up a pizza shop

Shortest Paths

All-Pairs Shortest Path – Floyd's Algorithm

- Find the diameter of a graph
  - The **longest shortest-path distance over all pairs of vertices**
  - e.g., longest possible time for a network packet to be delivered
- Both examples require computation of the **shortest path between all pairs of vertices** in a given graph (n x n distance matrix)

Shortest Paths

All-Pairs Shortest Path – Floyd's Algorithm

- Simple solution: call Dijkstra's algorithm from each of the n possible starting vertices (hence n x n distance matrix)

- Floyd-Warshall Algorithm

  - Construct n x n shortest-path distance matrix directly from the original n x n weight matrix

  - Use adjacency matrix instead of adjacency list data structure

Shortest Paths

All-Pairs Shortest Path – Floyd's Algorithm

Use adjacency matrix instead of adjacency list data structure

```
typedef struct {
  int weight[MAXV+1][MAXV+1];  /* adjacency/weight info       */
  int nvertices;               /* number of vertices in graph */
} adjacency_matrix;
```

Initialize each non-edge to `MAXINT` `(INT_MAX)` instead of zero

Will flag non-edge and still be ignored in shortest path algorithm

Shortest Paths

Floyd-Warshall Algorithm

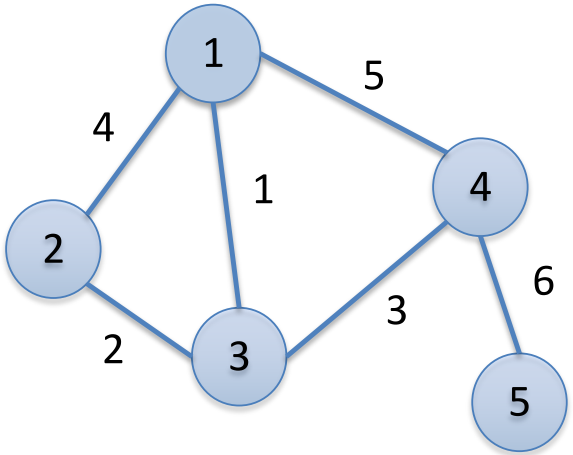
- Number all the vertices from 1 to n
  - Use these numbers to order the vertices (not label them)
- Define $W[i, j]^k$ to be the length of the shortest path from i to j

  using vertices numbered from $1, 2, \ldots, k$

  as possible intermediate vertices

Shortest Paths

Floyd-Warshall Algorithm



∞	4	1	5	∞
4	∞	2	∞	∞
1	2	∞	3	∞
5	∞	3	∞	6
∞	∞	∞	6	∞

Shortest Paths

Floyd-Warshall Algorithm

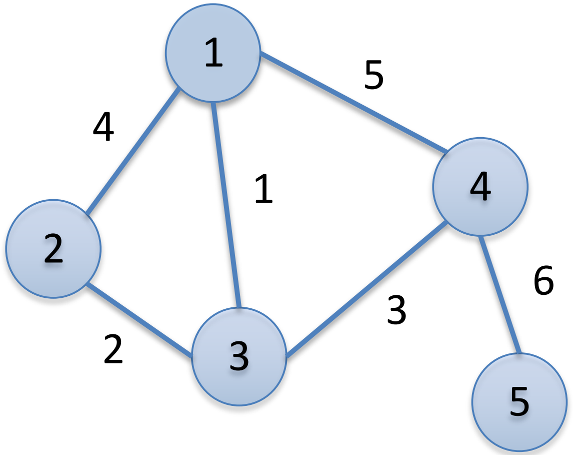
- Perform k iterations
- Iteration k allows only the first k vertices as possible intermediate steps on the path from between each pair of vertices x and y
- With each iteration we allow a richer set of possible shortest paths by adding a new vertex as a possible intermediary
- Allowing the k^{th} vertex as an intermediary stop helps only if there is a short path that goes through k

$$W[i, j]^k = \min(W[i, j]^{k\text{-}1}, W[i, k]^{k\text{-}1} + W[k, j]^{k\text{-}1})$$

Think about this …

Shortest Paths

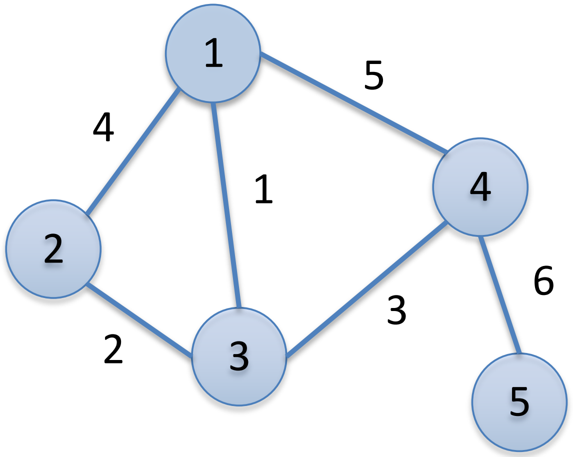
Floyd-Warshall Algorithm



$$\begin{matrix} \infty & 4 & 1 & 5 & \infty \\ 4 & \infty & 2 & \infty & \infty \\ 1 & 2 & \infty & 3 & \infty \\ 5 & \infty & 3 & \infty & 6 \\ \infty & \infty & \infty & 6 & \infty \end{matrix}$$

Shortest Paths

Floyd-Warshall Algorithm



k=1 (i.e., k is the intermediary vertex)

$$\begin{matrix} \infty & 4 & 1 & 5 & \infty \\ 4 & 8 & 2 & 9 & \infty \\ 1 & 2 & 2 & 3 & \infty \\ 5 & 9 & 3 & 10 & 6 \\ \infty & \infty & \infty & 6 & \infty \end{matrix}$$

$$W[i, j]^k = \min(W[i, j]^{k\text{-}1}, W[i, k]^{k\text{-}1} + W[k, j]^{k\text{-}1} \quad k=1$$

Shortest Paths

```
floyd(adjacency_matrix *g) {
   int i,j;          /* dimension counters           */
   int k;            /* intermediate vertex counter */
   int through_k;    /* distance through vertex k   */

   for (k=1; k<=g->nvertices; k++) {
      for (i=1; i<=g->nvertices; i++) {
         for (j=1; j<=g->nvertices; j++) {
            through_k = g->weight[i][k]+g->weight[k][j];
            if (through_k < g->weight[i][j]) {
                g->weight[i][j] = through_k;
            }
         }
      }
   }
}
```

Shortest Paths

Floyd-Warshall Algorithm

- $O(n^3)$
- No better than n calls to Dijkstra's algorithm
- Better in practice (tight loops)
- One of the few algorithms that work better on adjacency matrices
- Does not allow you to construct the actual shortest path between any given pair of vertices (not the point of the algorithm)