Data Structures and Algorithms for Engineers

Module 9: Complex Networks

Lecture 2: Communities (Part 1)

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Lecture DSA09-02

Complex Networks

- Communities
  - Fundamental Hypothesis & Connectedness and Density Hypothesis
  - Strong and weak communities
  - Graph partitioning & Community detection
    - Hierarchical clustering
    - Girvan-Newman Algorithm
    - Modularity
    - Random Hypothesis
    - Maximum Modularity Hypothesis
    - Greedy algorithm for community detection by maximizing modularity
  - Overlapping communities
    - Clique percolation algorithm and CFinder

This lecture is based on Chapter 9 of *Network Science* by A.-L. Barabási
(see https://networksciencebook.com/)

Complex Networks

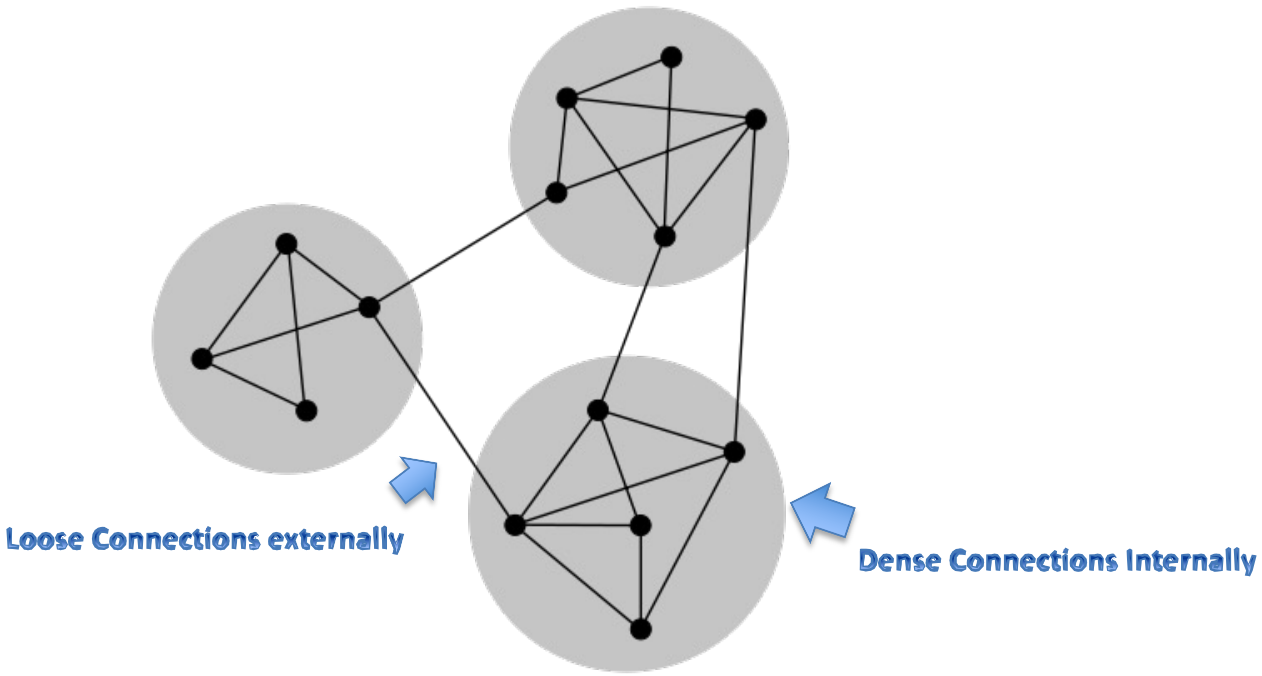
Communities

"In network science we call a *community* a group of nodes that have a higher likelihood of connecting to each other than to nodes from other communities."

L.A. Barabási

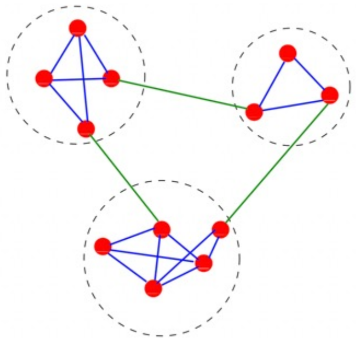
Complex Networks

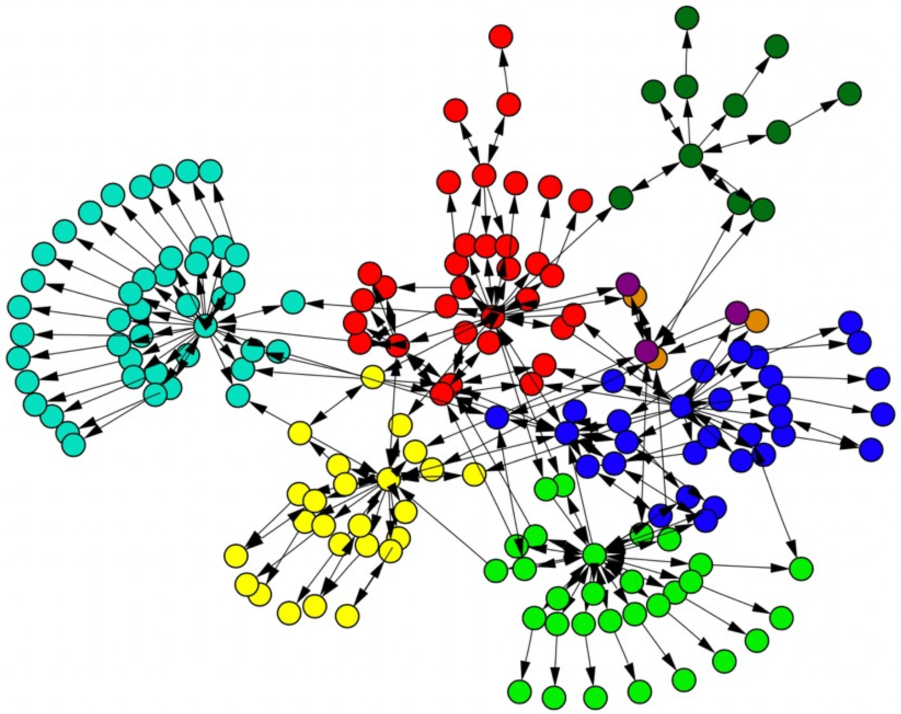
Communities



Complex Networks

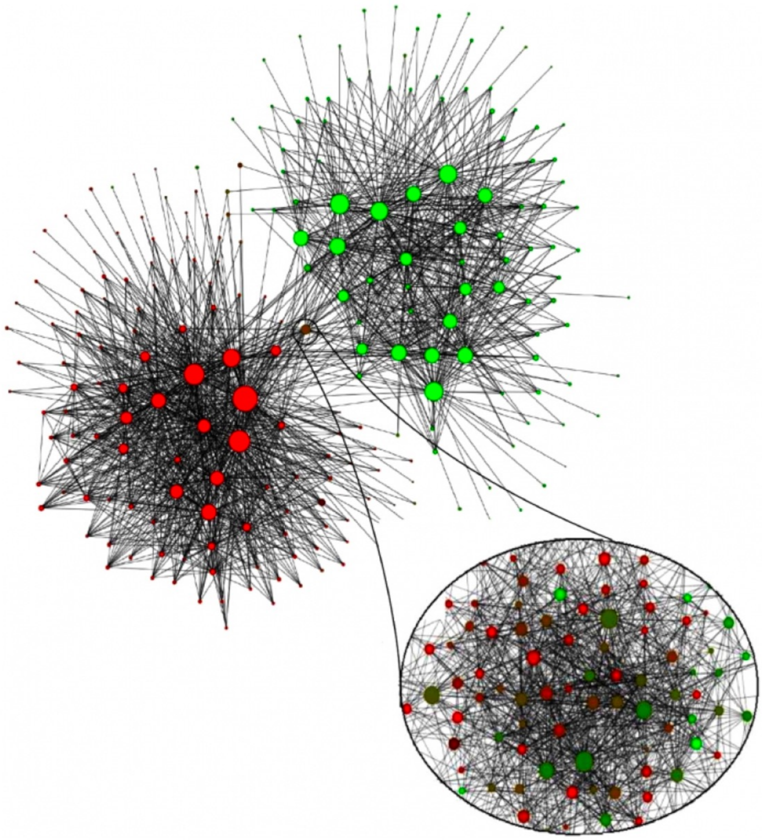
Communities





Complex Networks

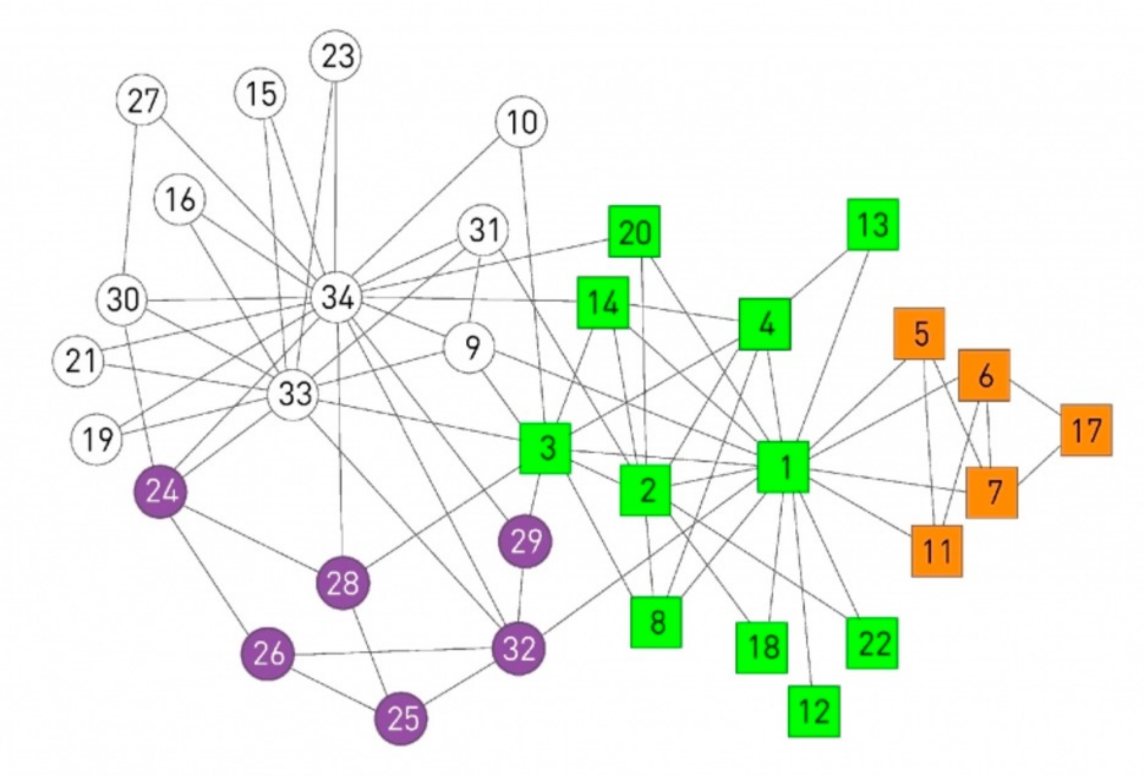
Communities



Communities in Belgium: red, French-speaking; green, Flemish-speaking
(node size = community size)

Complex Networks

Communities

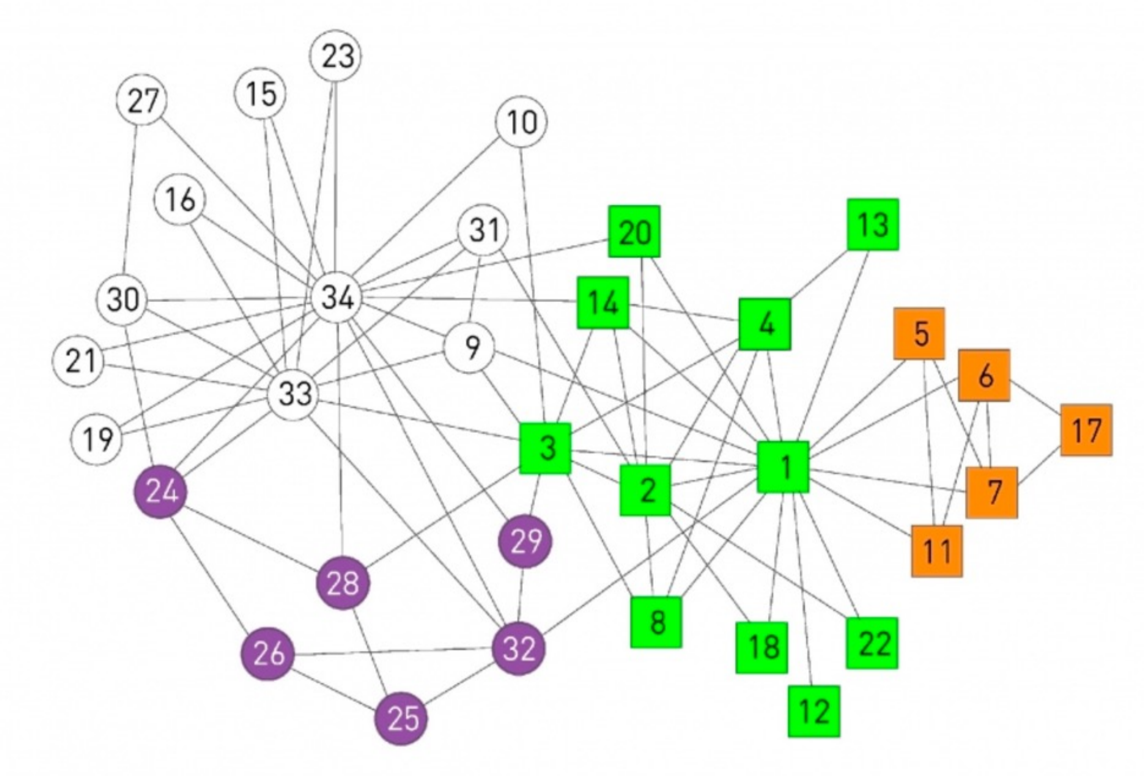


Zachary's Karate Club:

A conflict between the club's president and the instructor split the club into two.
About half of the members followed the instructor and the other half the president,
a breakup that unveiled the ground truth, representing club's underlying community structure

Complex Networks

Communities



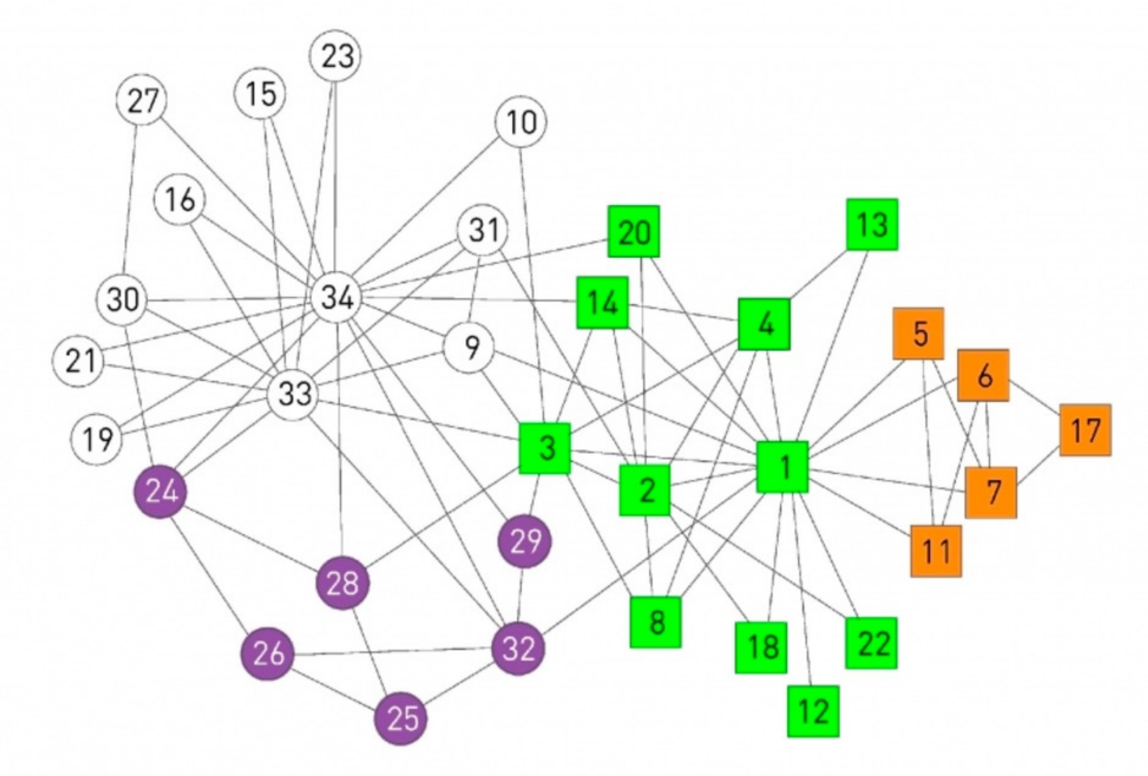
Zachary's Karate Club::

Links capture interactions between the club members *outside the club*.

The circles and the squares denote the two factions that emerged after the club split in two.

Complex Networks

Communities



Zachary's Karate Club:

The colors capture the best community partition predicted by an algorithm that optimizes the modularity coefficient

Complex Networks

Communities

H1: Fundamental Hypothesis

A network's community structure is uniquely encoded in its wiring diagram.

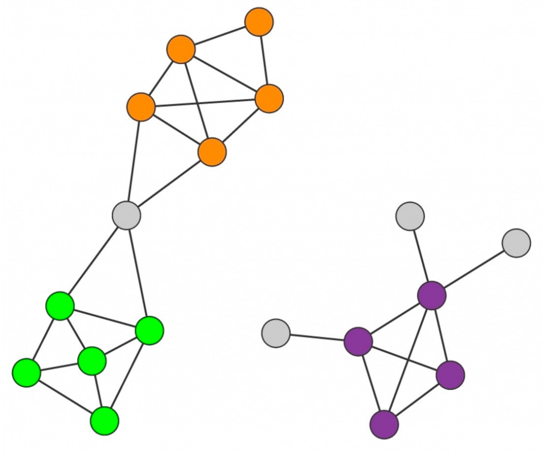
Complex Networks

Communities

H2: Connectedness and Density Hypothesis
A community is a locally dense connected subgraph in a network

Connected: all members of a community must be reached through other members of the same community

Dense: nodes that belong to a community have a higher probability to link to the other members of that community than to nodes that do not belong to the same community



Strong Community

C is a strong community if each node within C has more links within the community than with the rest of the graph

Specifically, a subgraph C forms a strong community if for each node $i \in C$,

$$k_i^{\mathrm{int}}(C) > k_i^{ext}(C)$$

Complex Networks

Communities

Weak Community

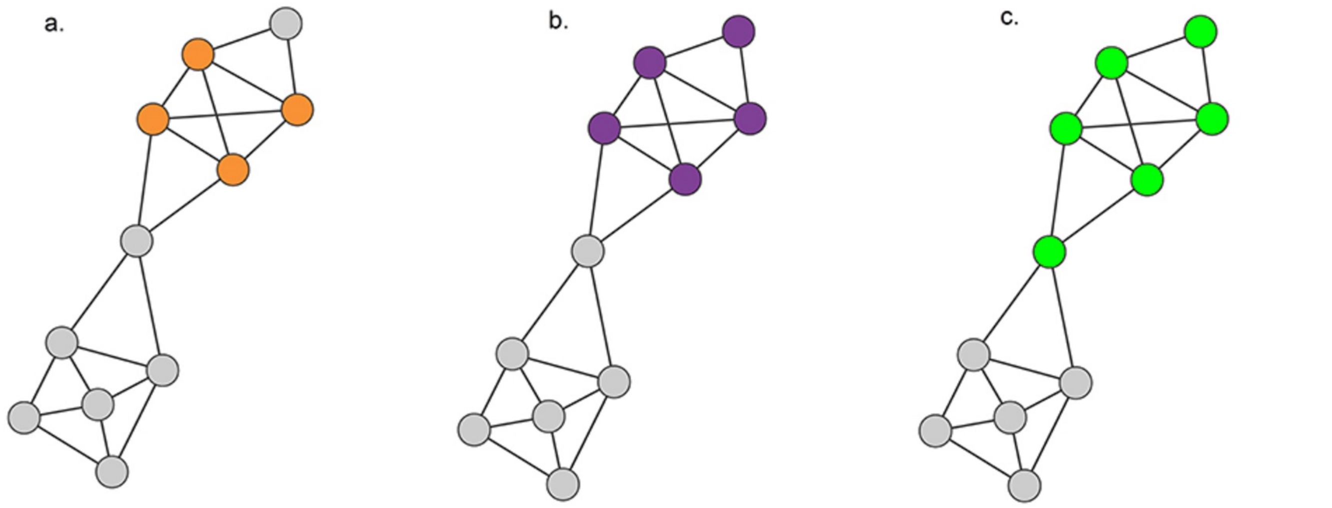
C is a *weak community* if the total internal degree of a subgraph exceeds its total external degree

Specifically, a subgraph C forms a weak community if

$$\sum_{i \in C} k_i^{\text{int}}(C) > \sum_{i \in C} k_i^{ext}(C)$$

Complex Networks

Communities



a. clique **b. strong community** **c. weak community**

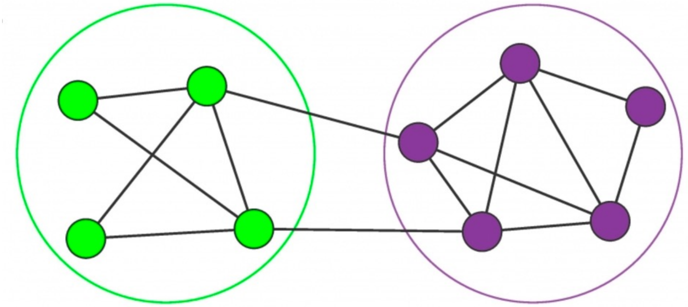
a *clique*
corresponds to a
complete subgraph
(rare)

Numbers of communities

How many ways can we group the nodes of a network into communities?

Graph partitioning, also called ***graph bisection***:

We aim to divide a network into two non-overlapping subgraphs,
such that the number of links between the nodes in the two groups, called the ***cut size***, is minimized



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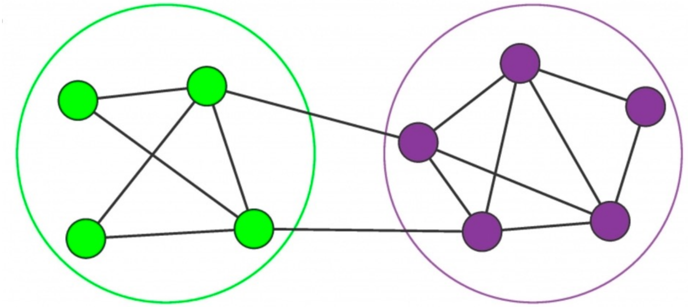
Communities

Numbers of communities

How many ways can we group the nodes of a network into communities?

Graph Bisection

Brute-force solution: inspect all possible divisions into two groups and choosing the one with the smallest cut size (exponential complexity)



Complex Networks

Communities

Graph partitioning vs. community detection

- Graph partitioning divides a network into a predefined number of smaller subgraphs
- Community detection aims to uncover the inherent community structure of a network

Complex Networks

Communities

Community detection

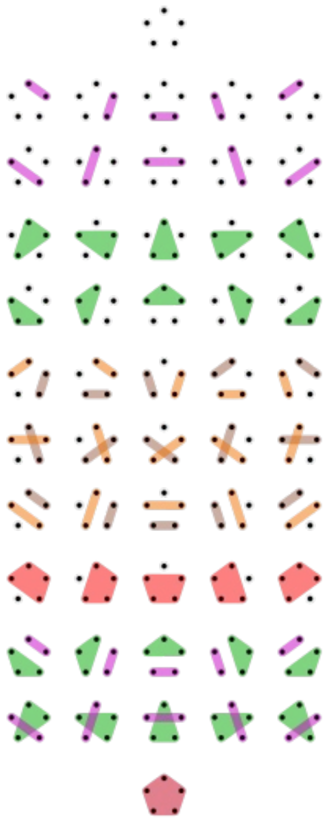
- Graph partitioning:
  the number and the size of communities is predefined

- Community detection:
  both parameters are unknown

- Idea: detect communities by investigating all possible partitions

The number of possible partitions is given by the Bell number $B_N = \frac{1}{e} \sum_{j=0}^{\infty} \frac{j^N}{j!}$

52 Partitions of a set with 5 elements



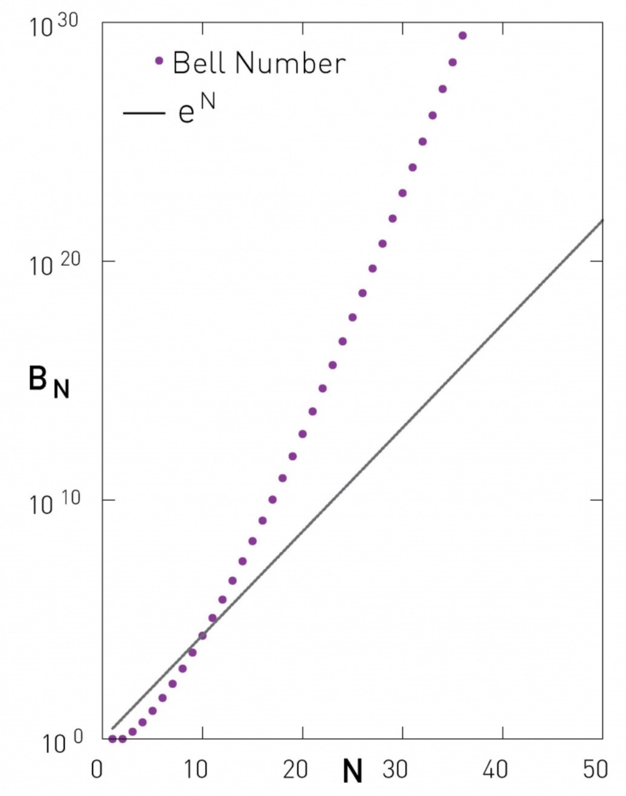
https://en.wikipedia.org/wiki/Bell_number

Complex Networks

Communities

Community detection

$$B_N = \frac{1}{e}\sum_{j=0}^{\infty}\frac{j^N}{j!}$$



Brute-force exponential-complexity algorithms that aim to identify communities by inspecting all possible partitions are computationally infeasible

Complex Networks

Communities

Community detection

We need polynomial-time algorithms that can uncover the community structure of large real networks …

Hierarchical Clustering

Brute-force exponential-complexity algorithms that aim to identify communities by inspecting all possible partitions are computationally infeasible

Complex Networks

Community Detection

Hierarchical Clustering

- Generate a similarity matrix x_{ij} indicating the similarity between vertex/node i and vertex/node j
- Iteratively identify groups of nodes with high similarity

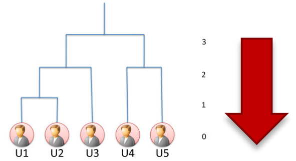
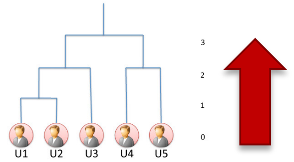
1. Agglomerative algorithms
   merge nodes with high similarity into the same community

2. Divisive algorithms
   isolate communities by removing low similarity links that tend to connect communities.

Both procedures generate a hierarchical tree, called a dendrogram, that predicts the possible community partitions

Complex Networks

Communities

Publication	Highlights	Example
Newman and Girvan (2004)	❑ Divisive Algorithm ❑ Remove the edge iteratively from the network	
Newman (2004)	❑ Agglomerative Algorithm ❑ Modularity: measure quality of communities	

Complex Networks

Community Detection

Divisive Procedures: the Girvan-Newman Algorithm

Step 1: Define Centrality

Step 2: Hierarchical Clustering

Complex Networks

Community Detection

Divisive Procedures: the Girvan-Newman Algorithm

Step 1: Define Centrality

The similarity matrix x_{ij} is called centrality and selects node pairs that are in different communities

x_{ij} is high if nodes i and j belong to different communities
x_{ij} is low if they are in the same community

Several options to choose from …

Complex Networks

Community Detection

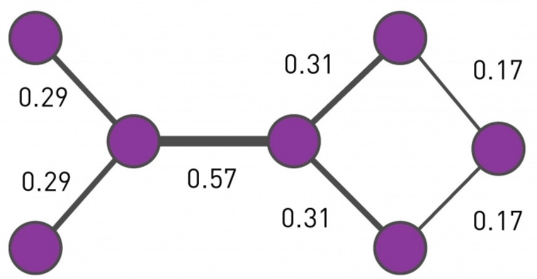
Divisive Procedures: the Girvan-Newman Algorithm

Step 1: Define Centrality

link betweenness

x_{ij} is defined as the number of shortest paths that go through the link (i, j)

Links connecting different communities are expected to have large x_{ij} while links within a community have small x_{ij}



NB: these link betweenness values are based on a single shortest path between two nodes
(which is not what the Girvan-Newman algorithm stipulates)

Complex Networks

Community Detection

Divisive Procedures: the Girvan-Newman Algorithm

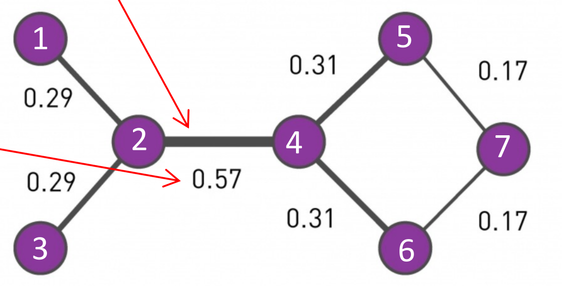
The similarity matrix for a network with n nodes has n^2 entries
However, n of these don't count (these are the diagonal elements, i.e., the similarity of a node with itself)
so, there are $(n\ (n\text{-}1))$ relevant entries and, therefore, $(n\ (n\text{-}1))/2$ shortest paths; remember that the network is undirected and unweighted and so the shortest path from node *i* to node *j* is the same as from node *j* to node *i*.

For the network below, there 7 nodes and 21 shortest paths.

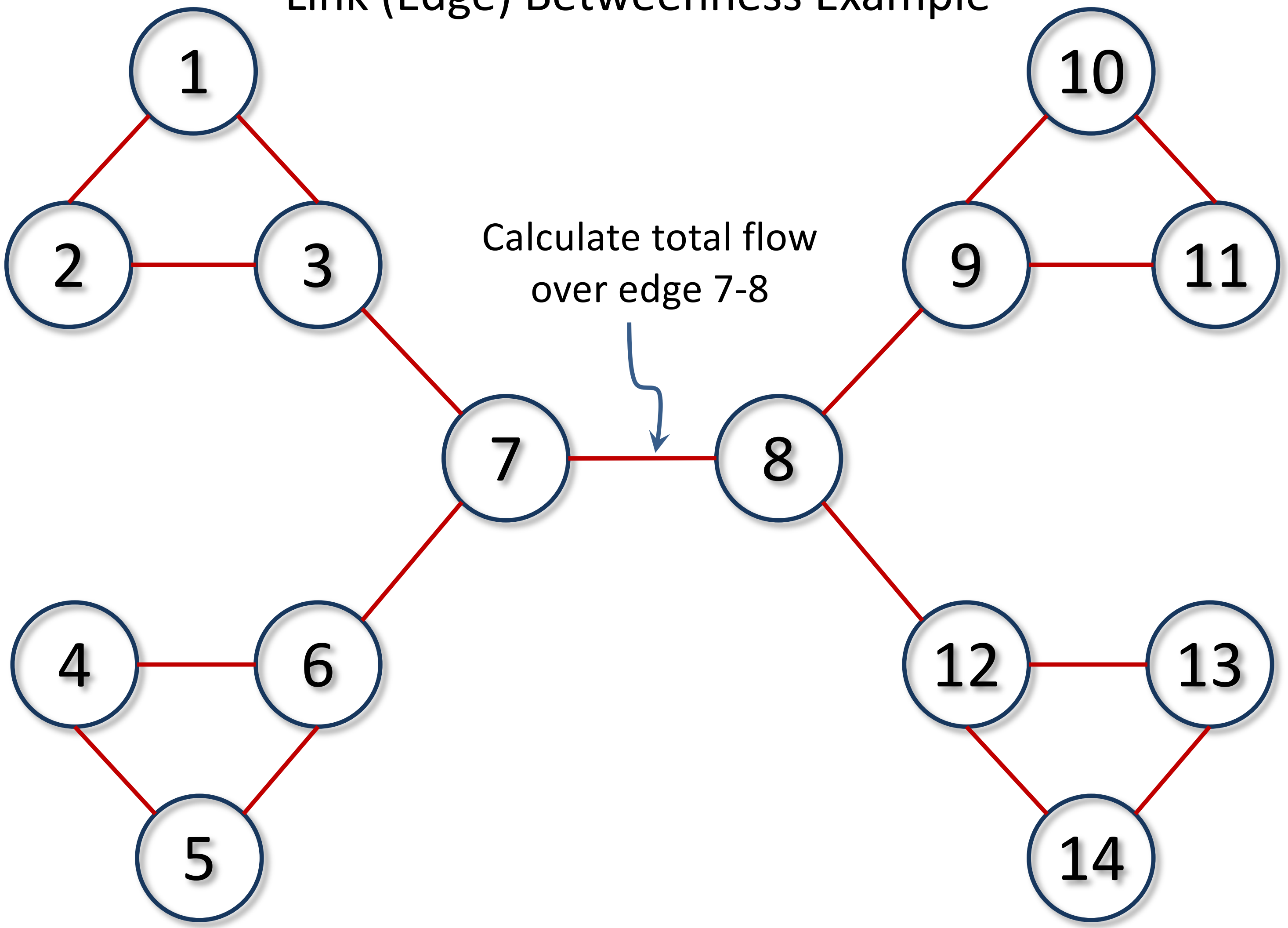
There are 12 shortest paths with this link: (1, 2, 4, 5, 7), (1, 2, 4, 5), : (1, 2, 4, 6), (1, 2, 4)
(3, 2, 4, 5, 7), (3, 2, 4, 5), (3, 2, 4, 6), (3, 2, 4),
(2, 4, 5, 7), (2, 4, 5), (2, 4, 6), (2, 4)

The link inbetweenness value is
12/21 = 0.57

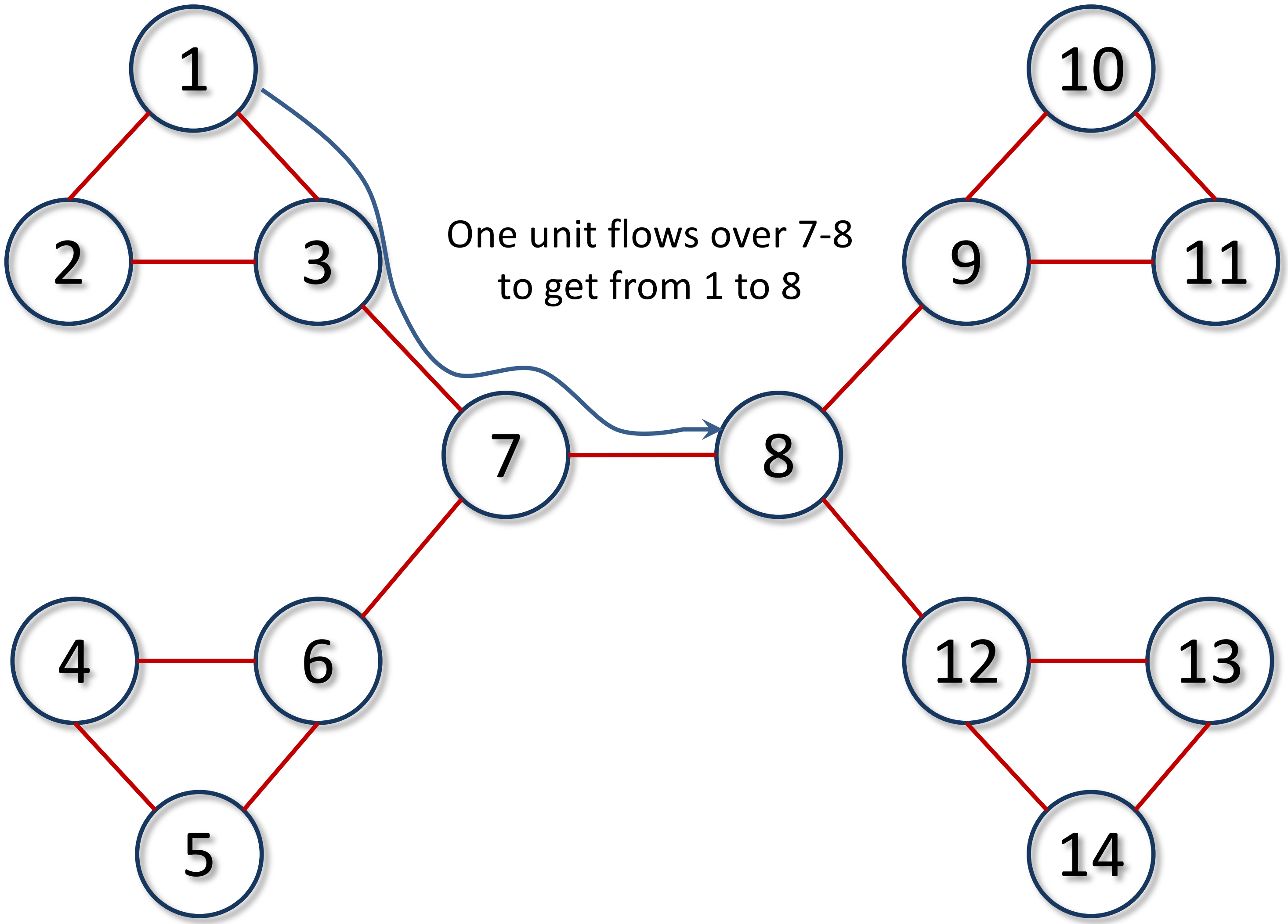
The other inbetweeness values can be calculated in a similar manner

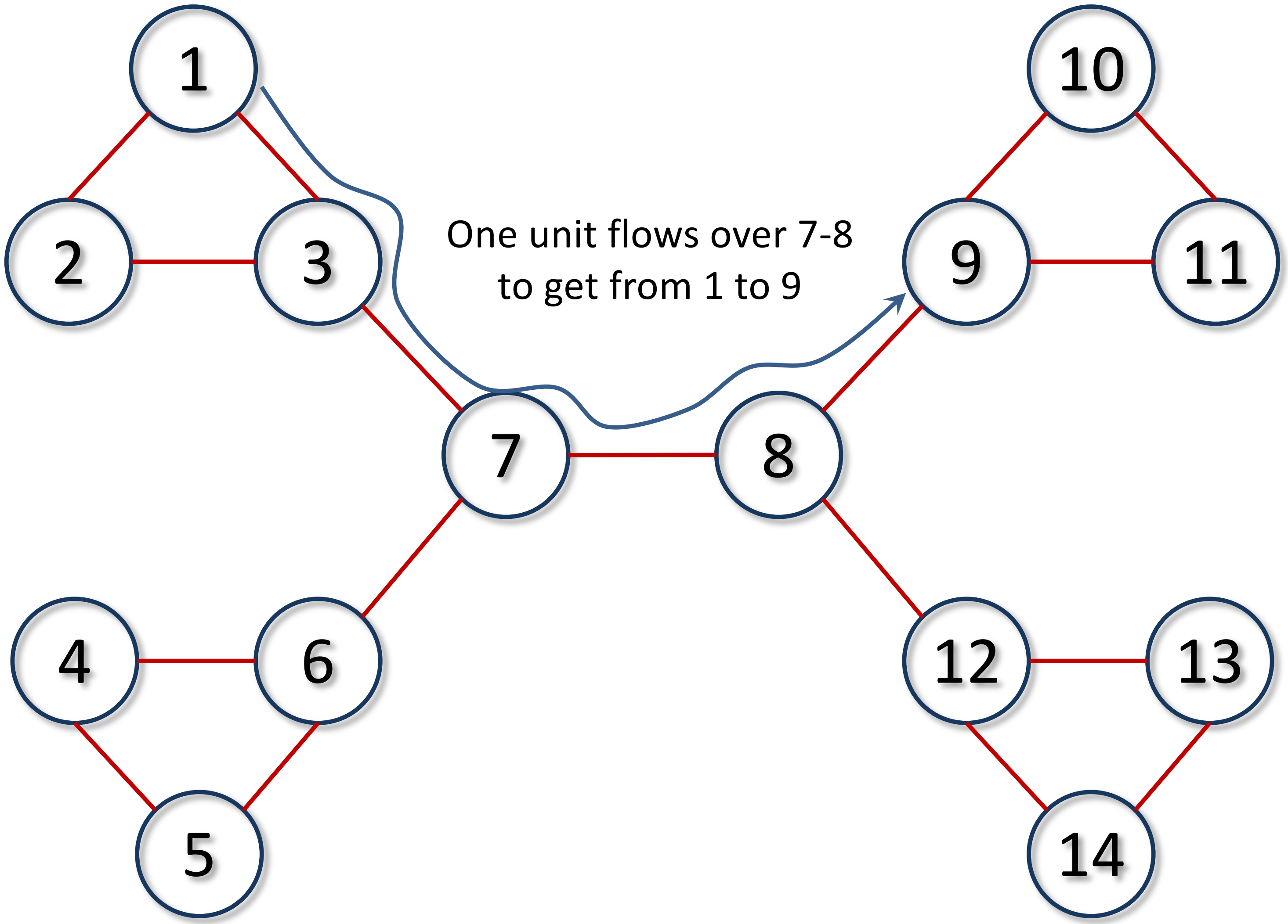


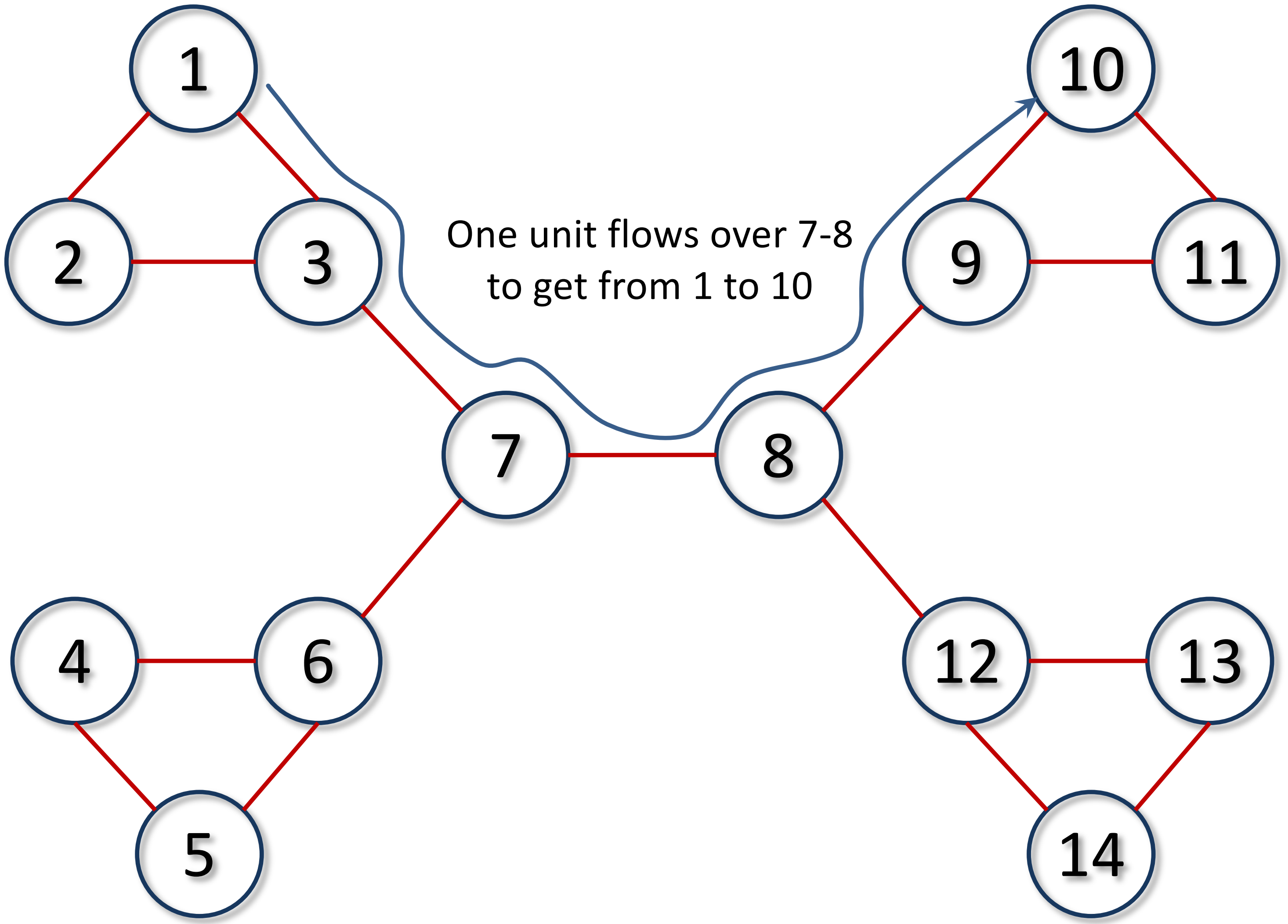
Link (Edge) Betweenness Example

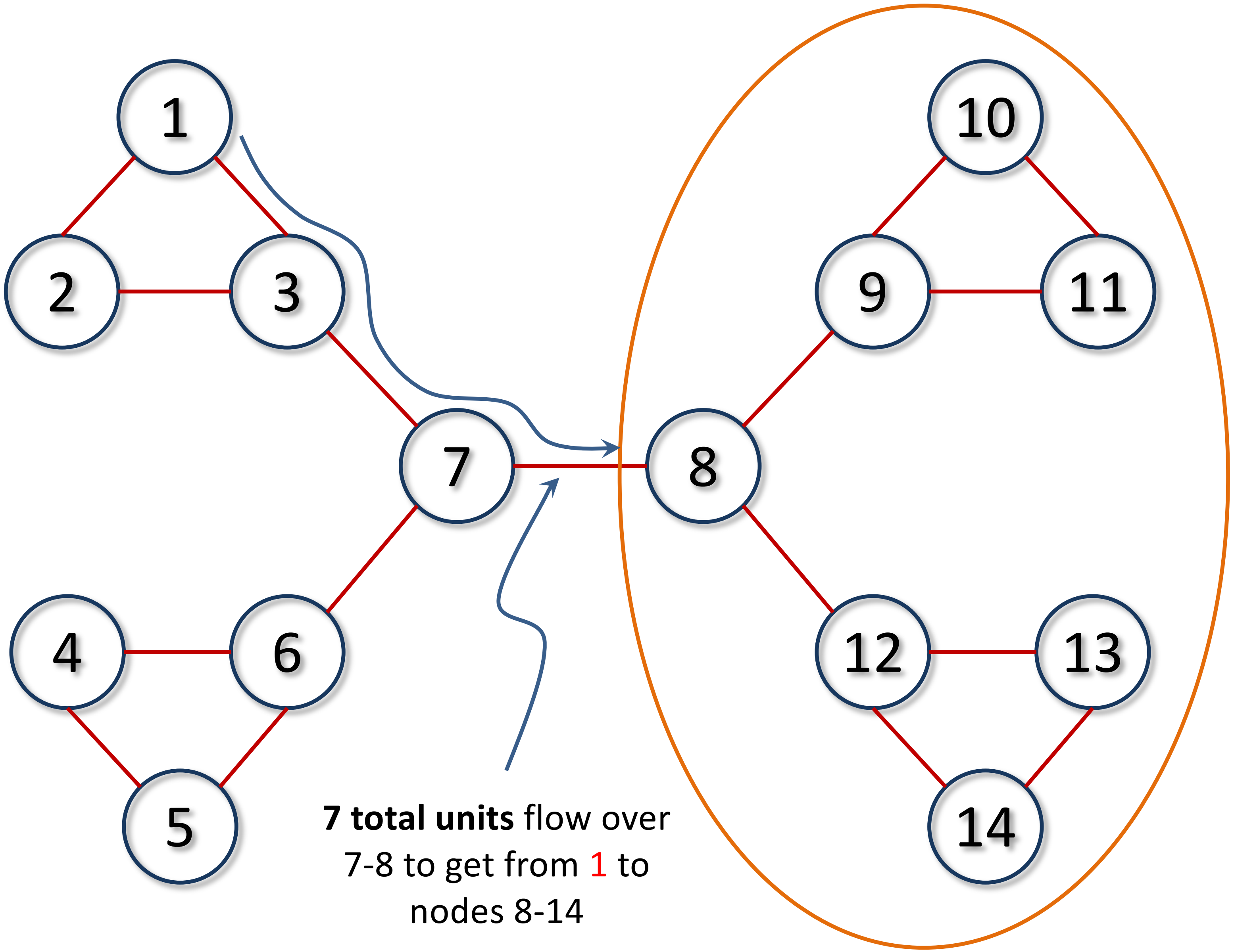


Credit: Frank McCown, Intro to Web Science, Harding University

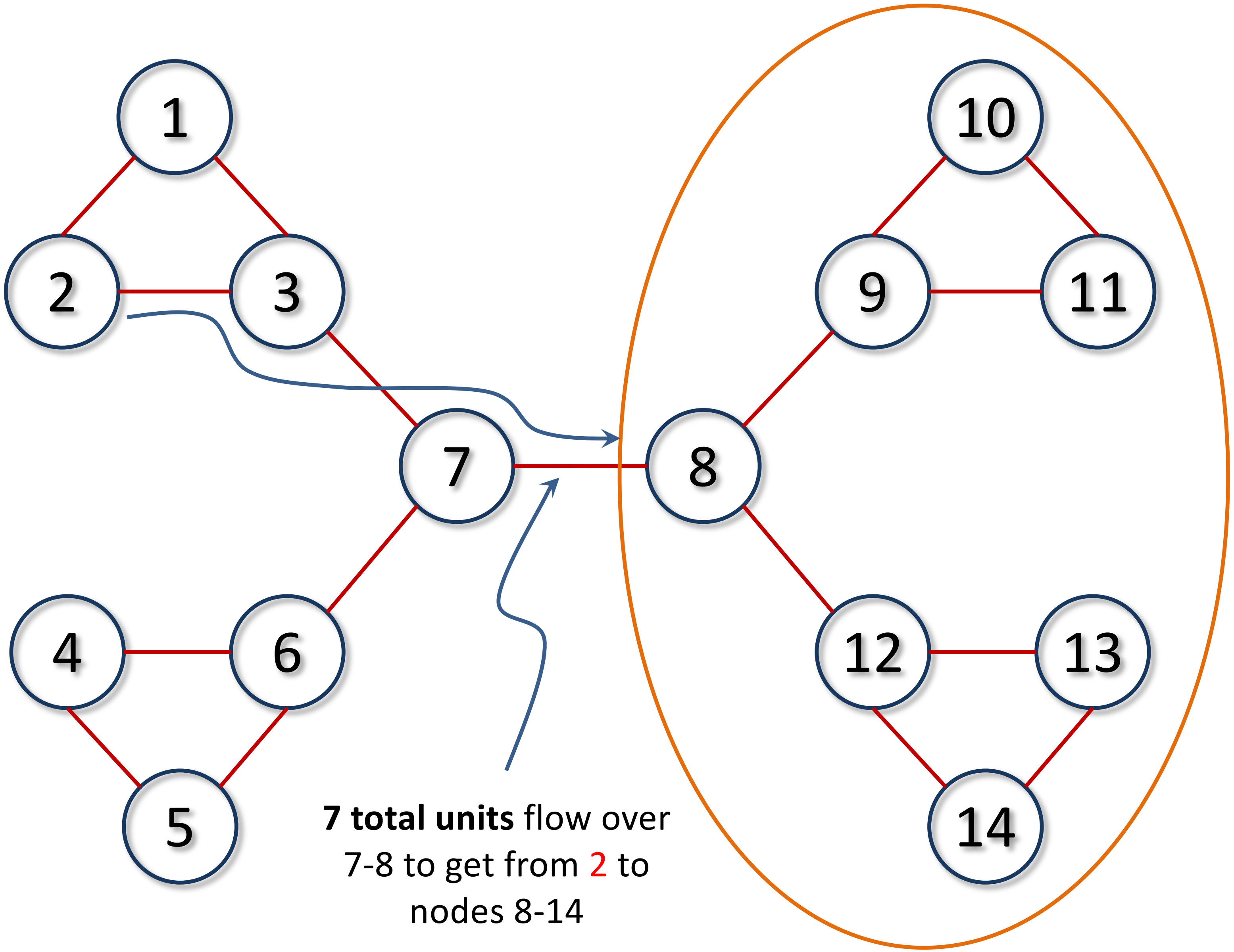

1
2
3
4
5
6
7
8
9
10
11
12
13
14
One unit flows over 7-8
to get from 1 to 8


1
2
3
7
8
9
10
11
4
6
5
12
13
14
One unit flows over 7-8
to get from 1 to 9

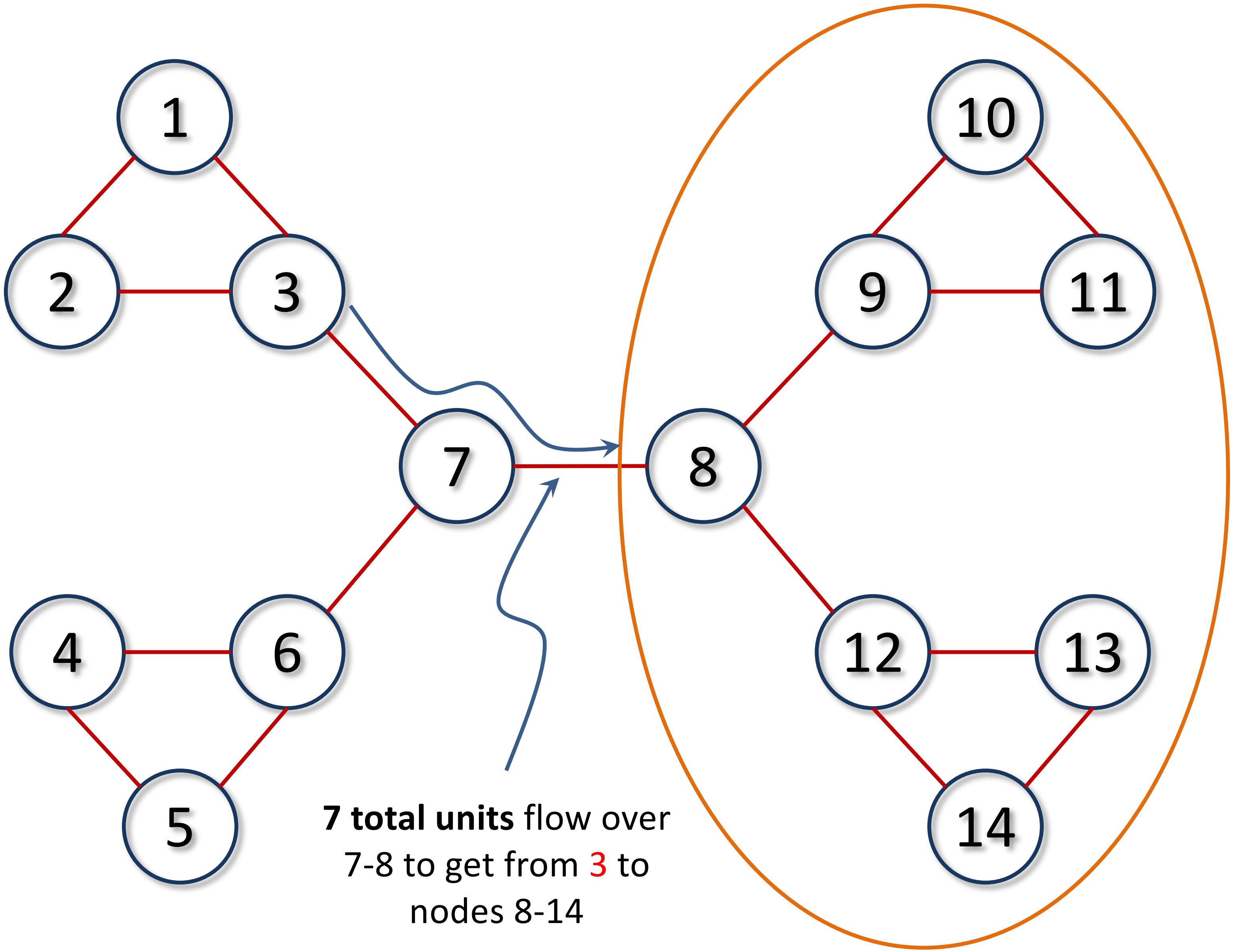

1
2
3
4
5
6
7
8
9
10
11
12
13
14
One unit flows over 7-8
to get from 1 to 10

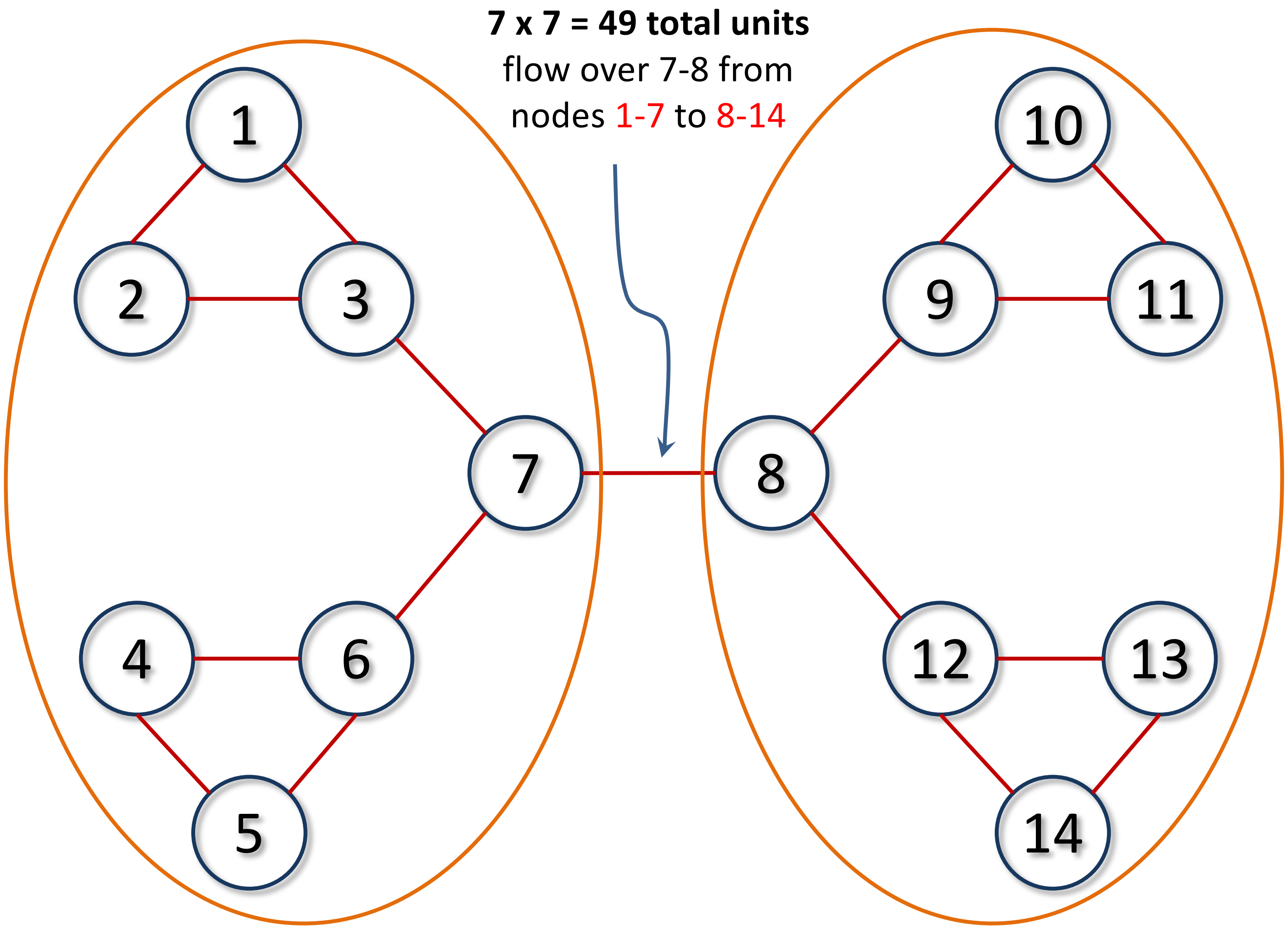


7 total units flow over 7-8 to get from 1 to nodes 8-14

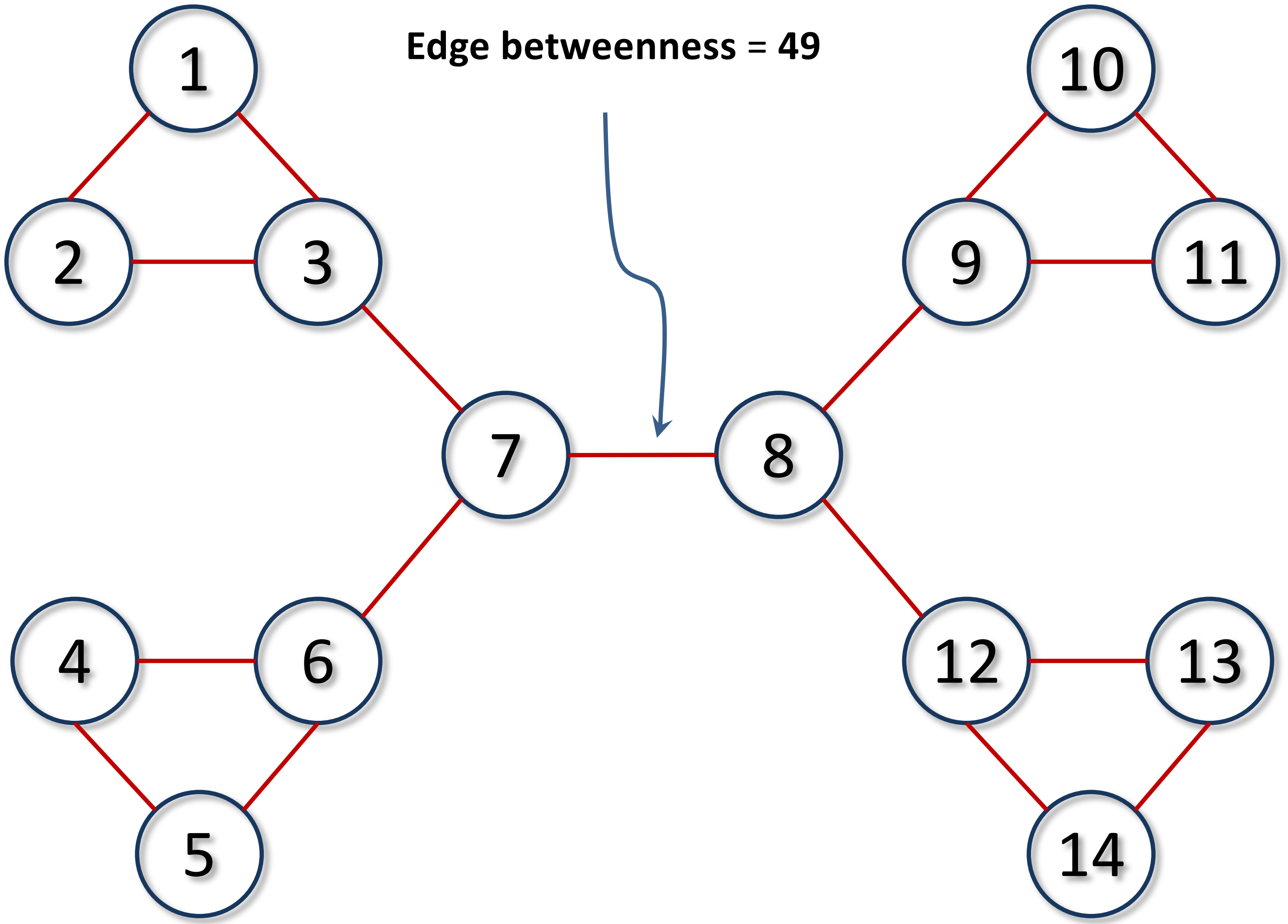


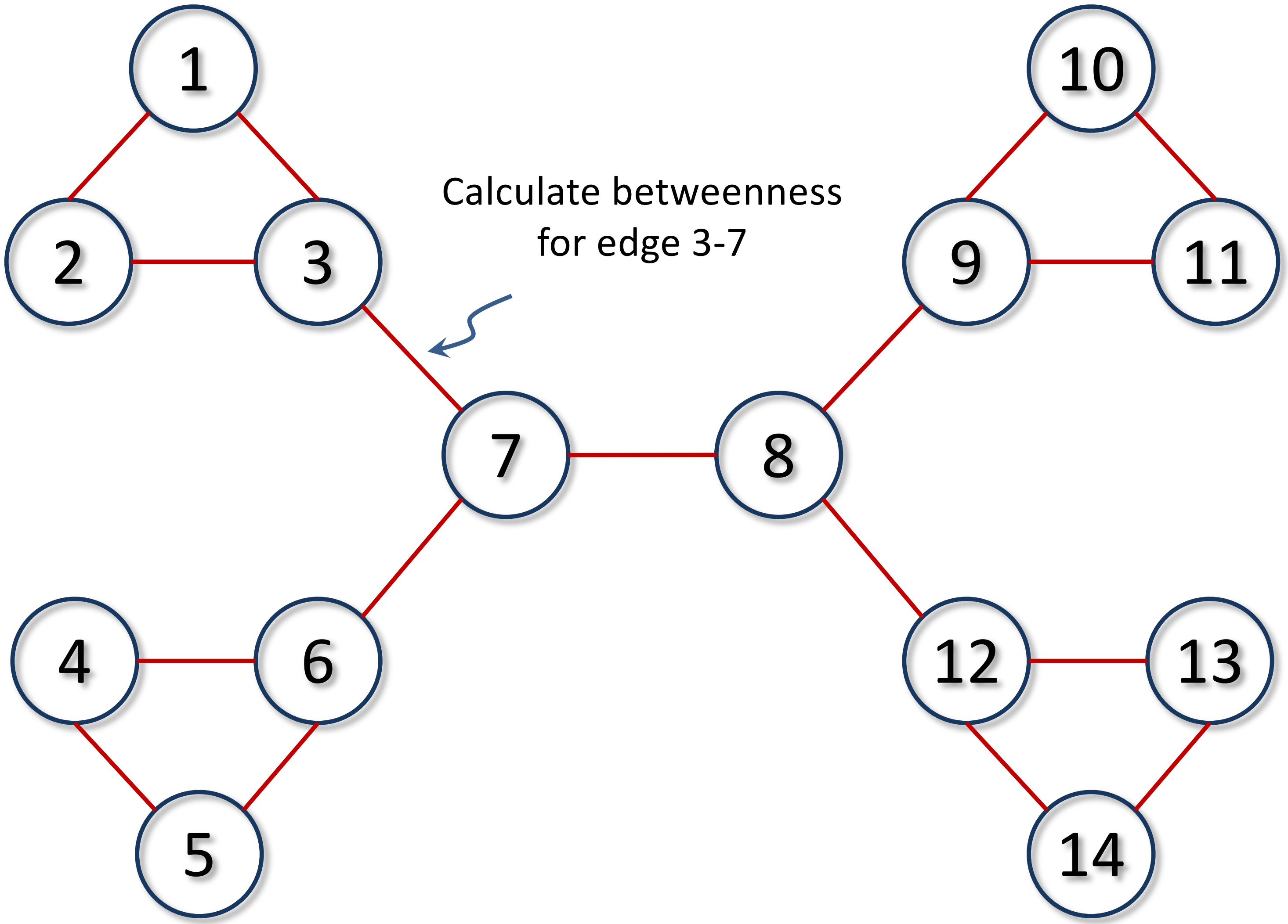
7 total units flow over 7-8 to get from 2 to nodes 8-14

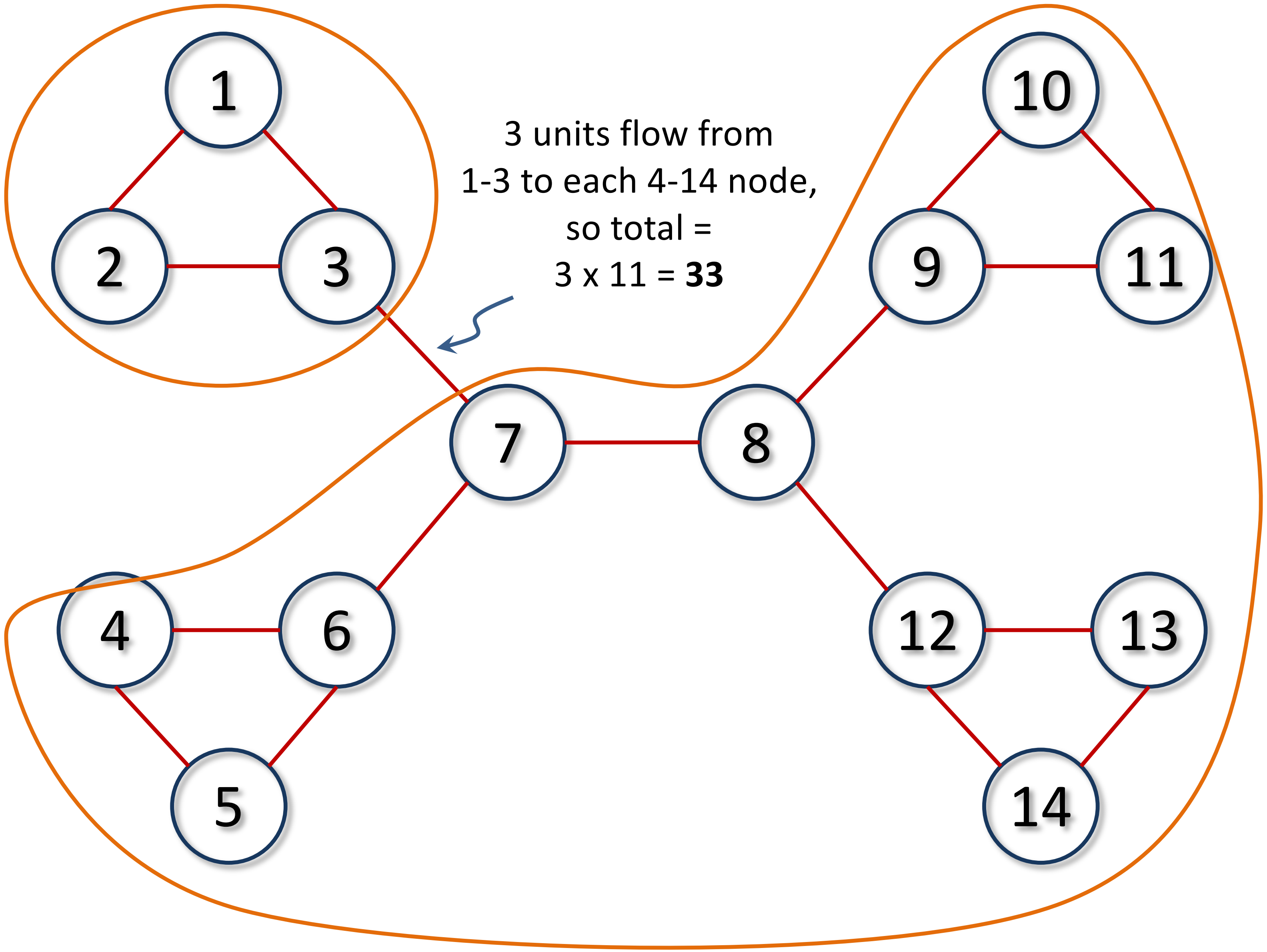

1
2
3
4
5
6
7
8
9
10
11
12
13
14
7 total units flow over 7-8 to get from 3 to nodes 8-14

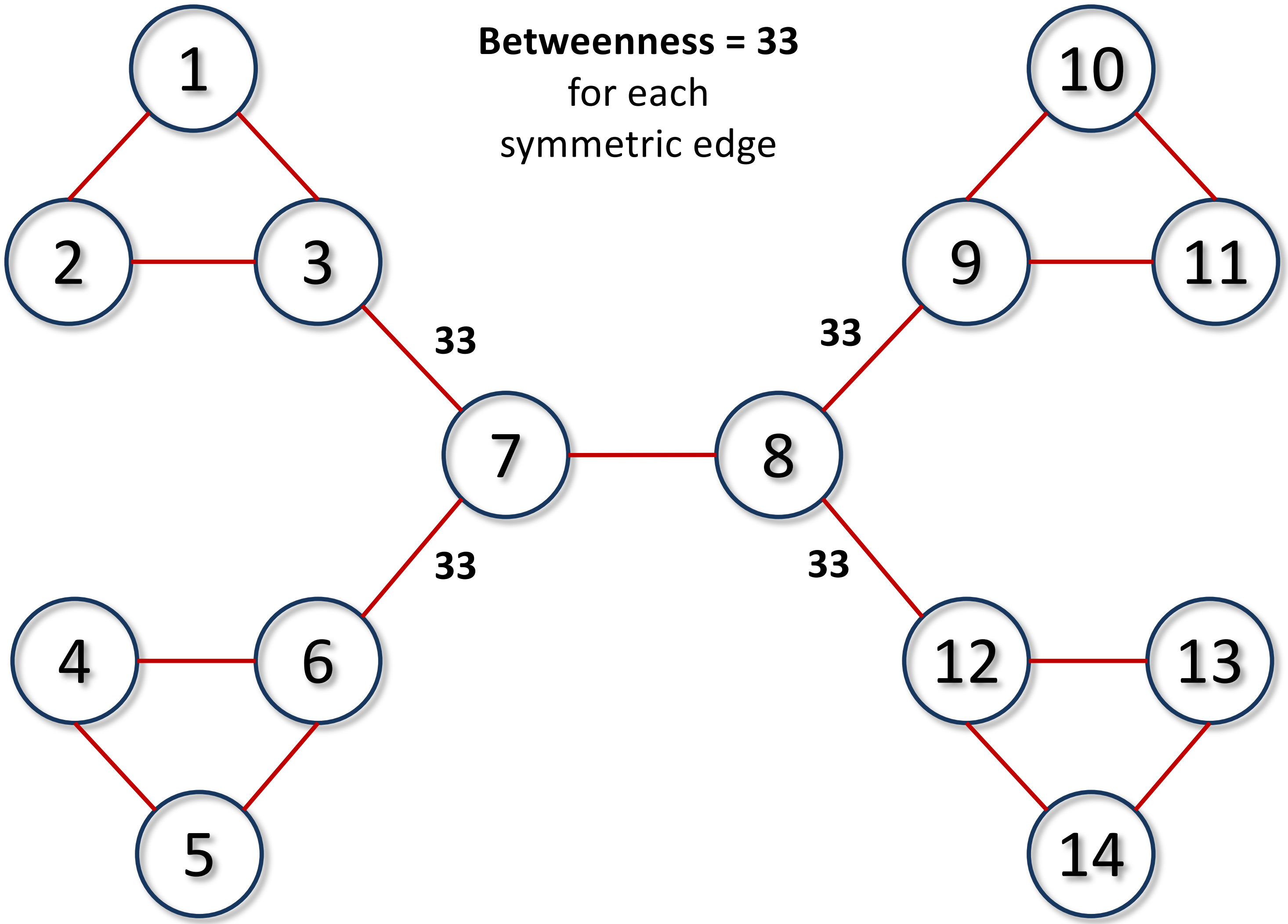


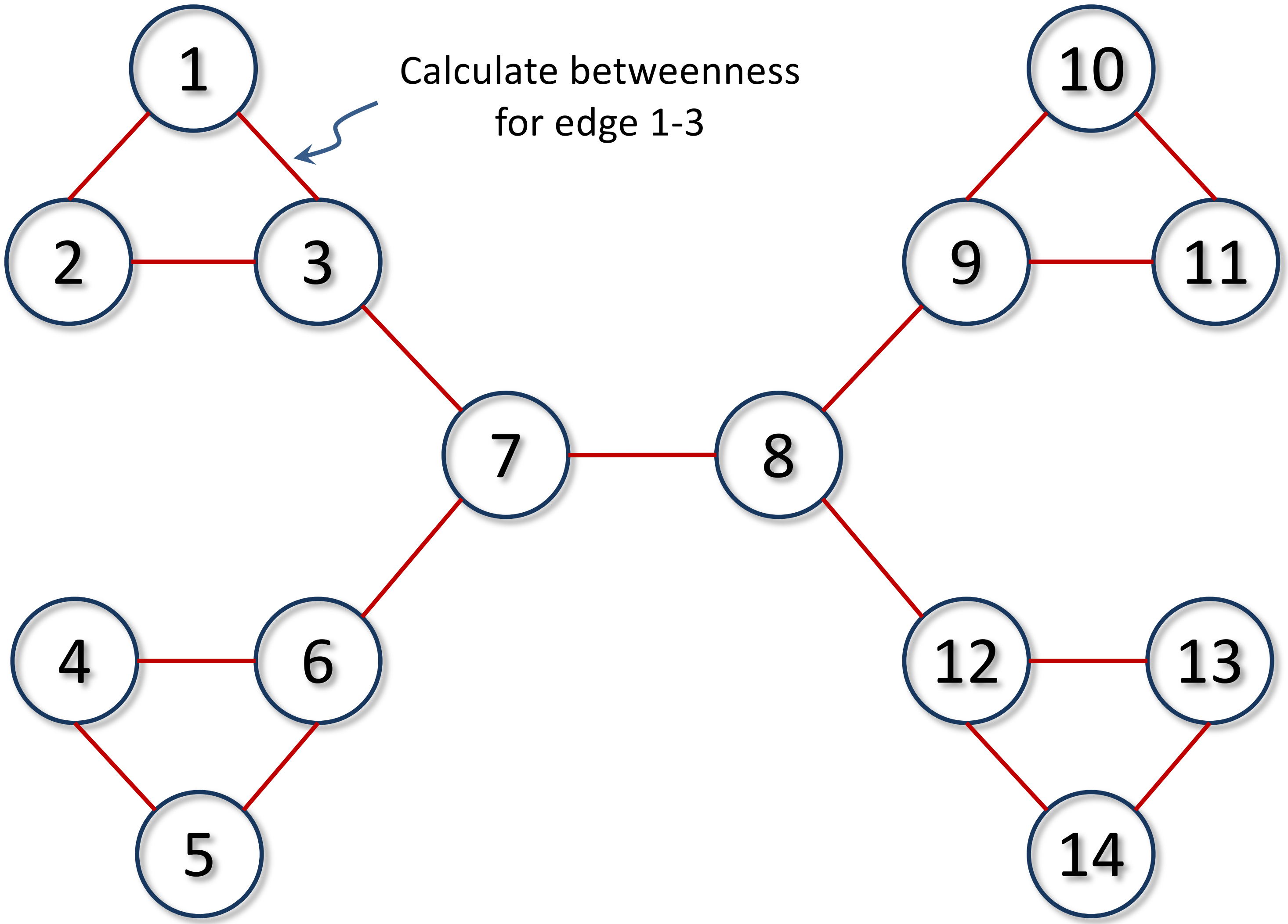
7 x 7 = 49 total units
flow over 7-8 from
nodes 1-7 to 8-14
1
2
3
4
5
6
7
8
9
10
11
12
13
14

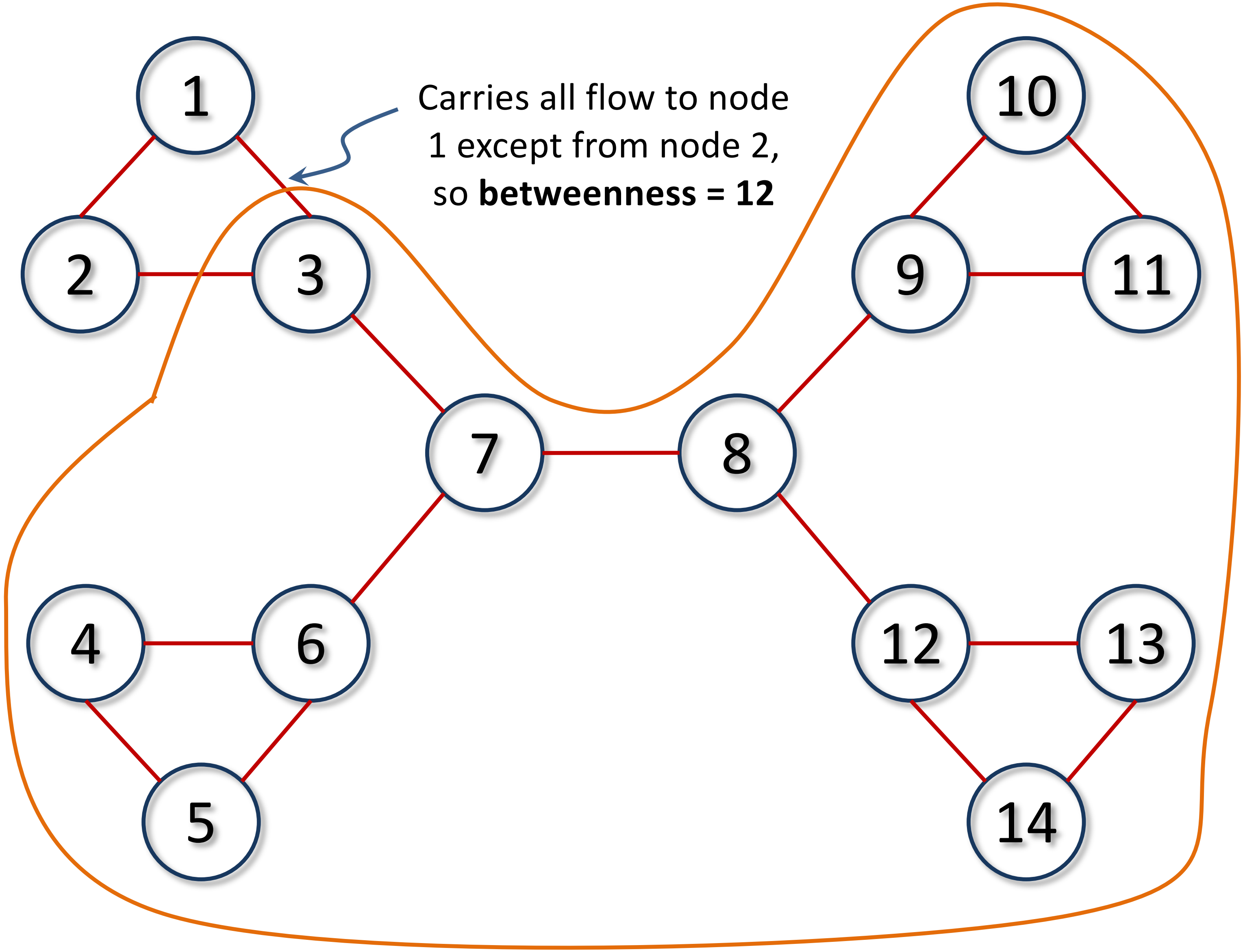

Edge betweenness = 49
1
2
3
4
5
6
7
8
9
10
11
12
13
14

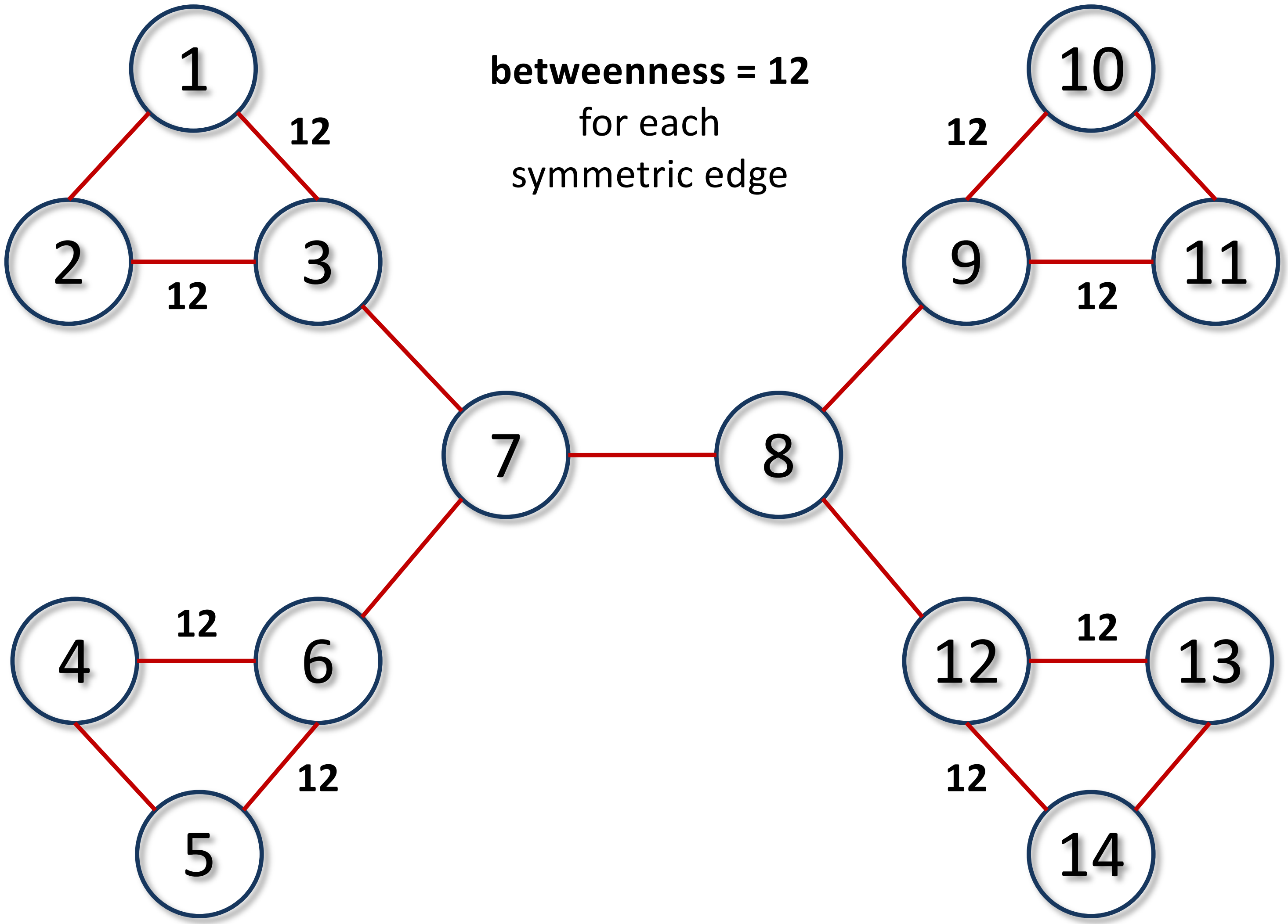

1
2
3
4
5
6
7
8
9
10
11
12
13
14
Calculate betweenness
for edge 3-7

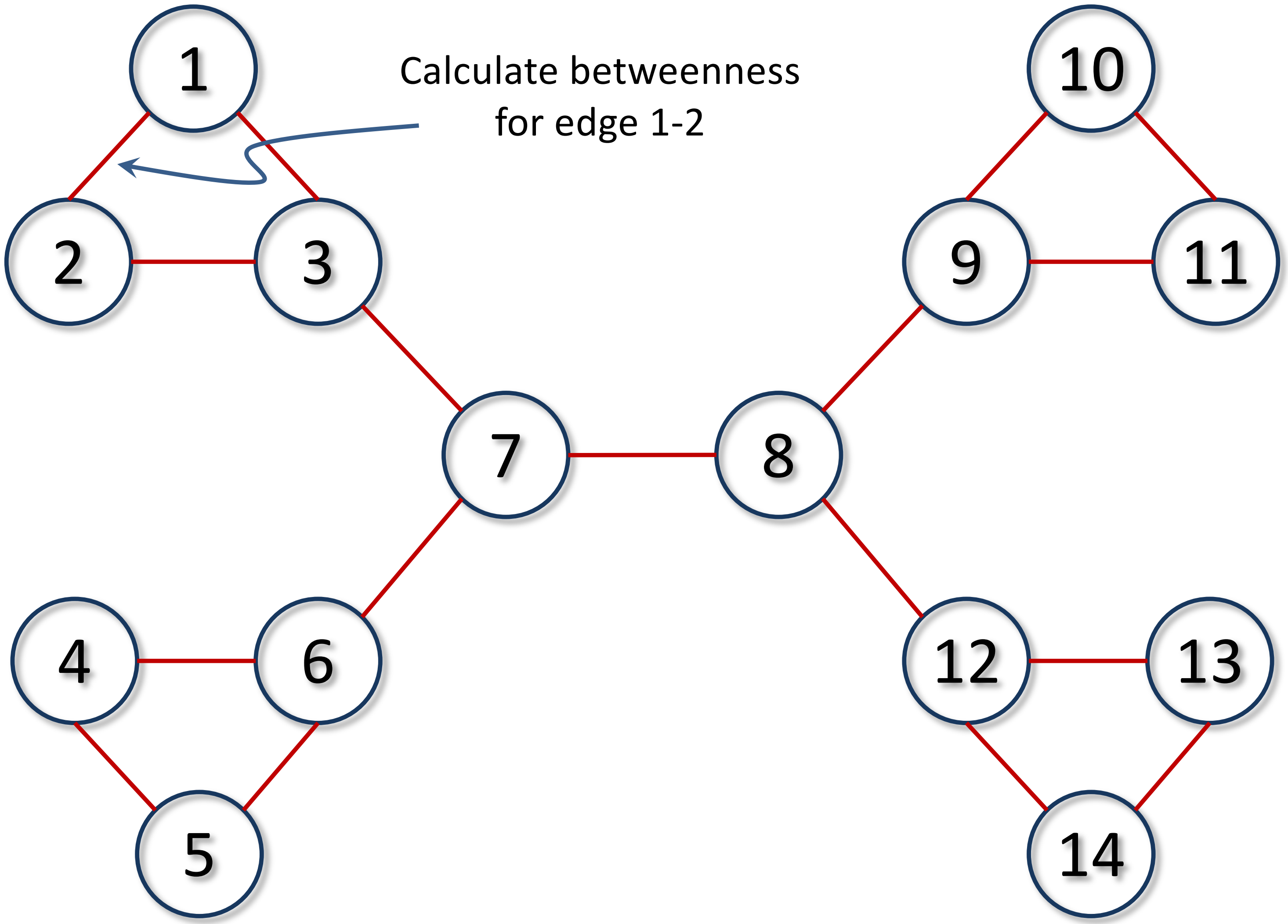

1
2
3
3 units flow from
1-3 to each 4-14 node,
so total =
3 x 11 = **33**
10
9
11
7
8
4
6
5
12
13
14


Betweenness = 33
for each
symmetric edge
1
2
3
4
5
6
7
8
9
10
11
12
13
14
33
33
33
33

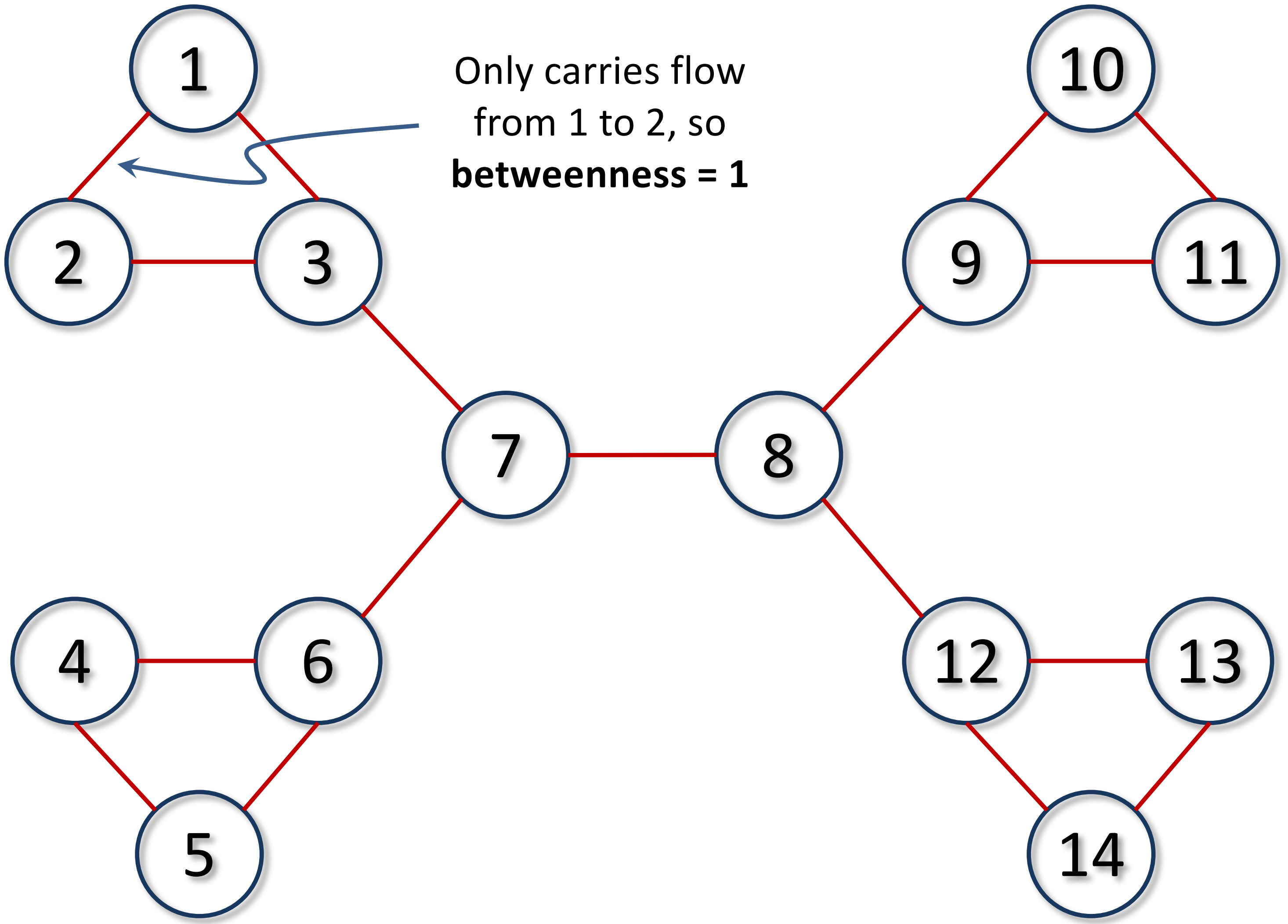

Calculate betweenness
for edge 1-3
1
2
3
4
5
6
7
8
9
10
11
12
13
14


1
2
3
4
5
6
7
8
9
10
11
12
13
14
Carries all flow to node 1 except from node 2, so **betweenness = 12**

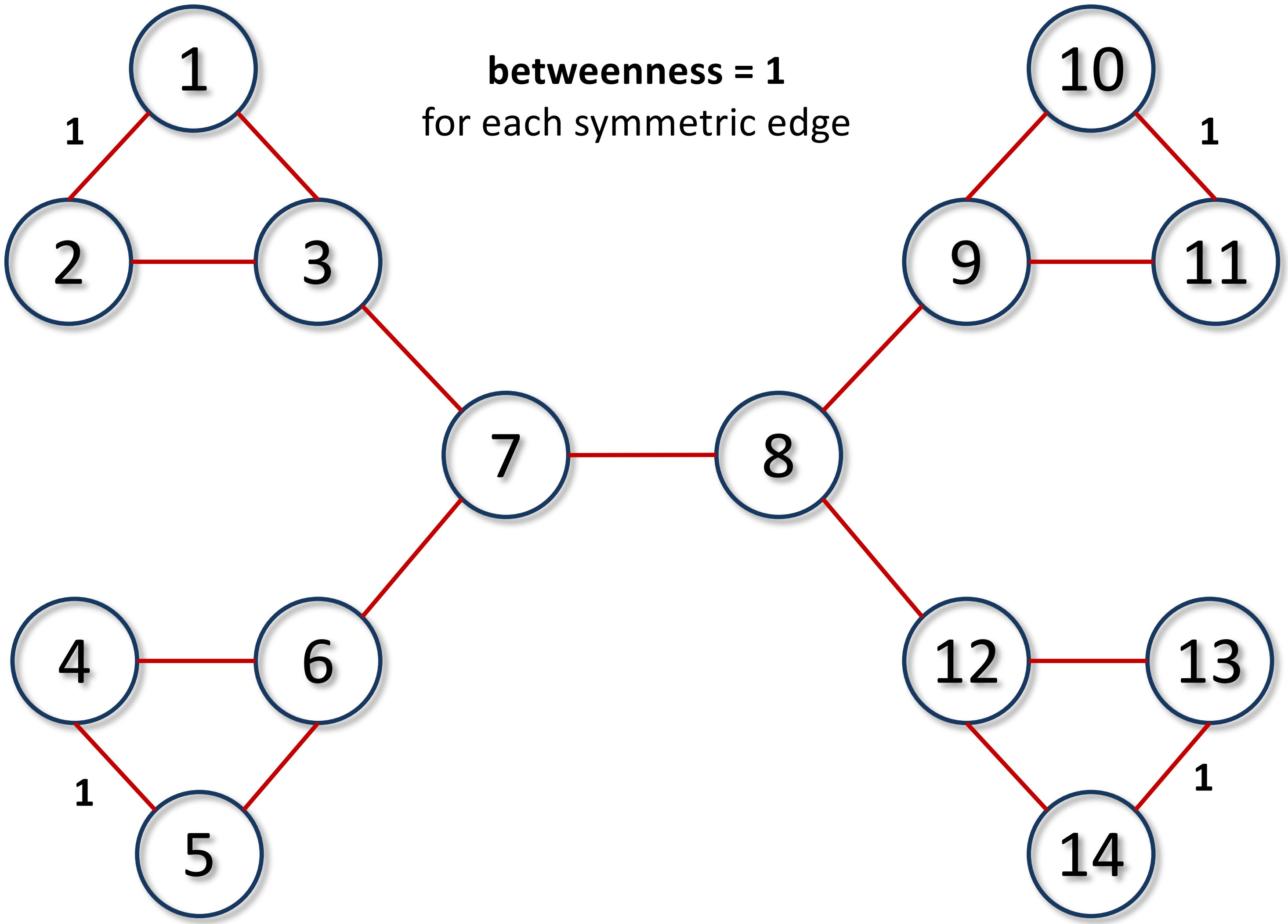

betweenness = 12
for each
symmetric edge
1
2
3
4
5
6
7
8
9
10
11
12
13
14
12
12
12
12
12
12
12
12

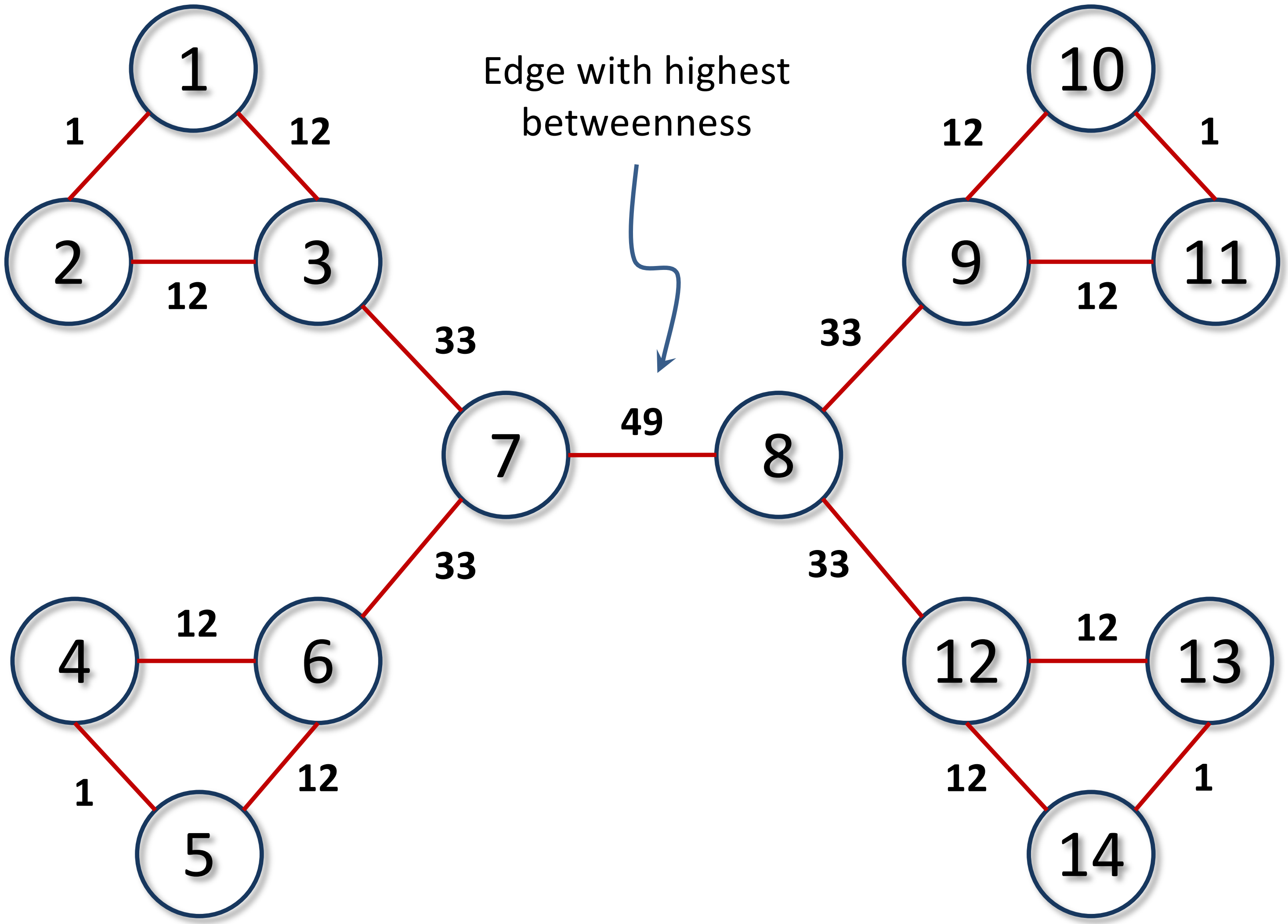


Calculate betweenness
for edge 1-2


1
2
3
4
5
6
7
8
9
10
11
12
13
14
Only carries flow from 1 to 2, so **betweenness = 1**

betweenness = 1
for each symmetric edge




Edge with highest betweenness
1
2
3
4
5
6
7
8
9
10
11
12
13
14
1
12
12
33
33
49
33
33
12
1
12
12
12
1
12
1
12
12

Complex Networks

Community Detection

Divisive Procedures: the Girvan-Newman Algorithm

Step 2: Hierarchical Clustering

1. Compute the centrality x_{ij} of each link

2. Remove the link with the largest centrality.
   In case of a tie, choose one link randomly

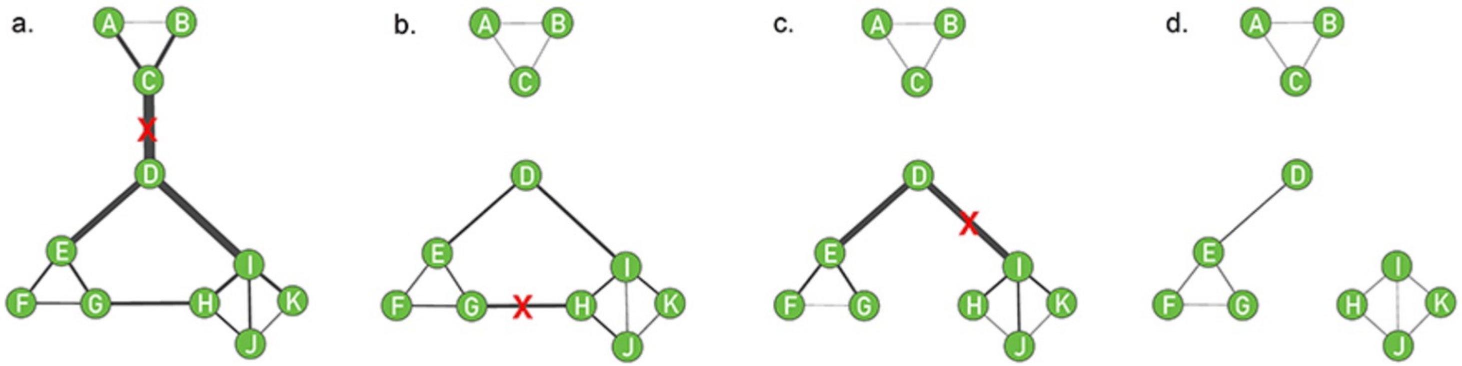
3. Recalculate the centrality of each link for the altered network

4. Repeat steps 2 and 3 until all links are removed

Complex Networks

Community Detection

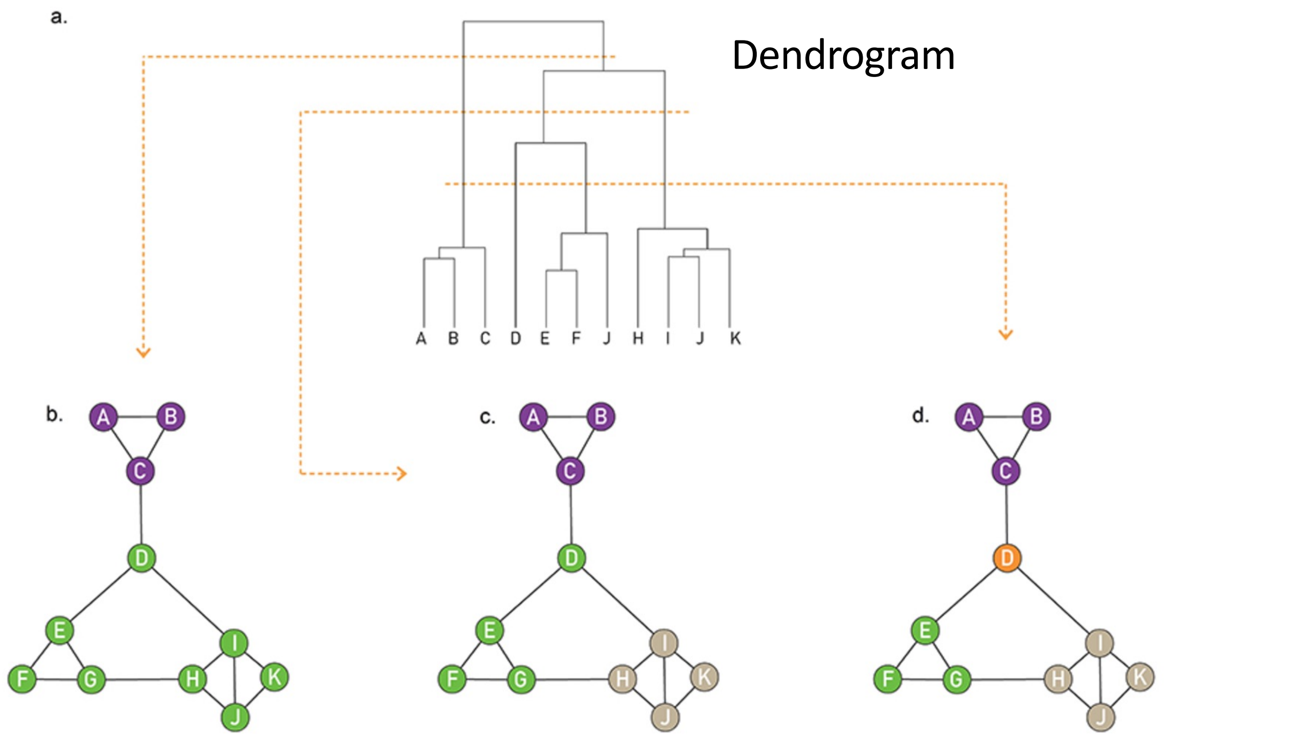
Divisive Procedures: the Girvan-Newman Algorithm



Complex Networks

Community Detection

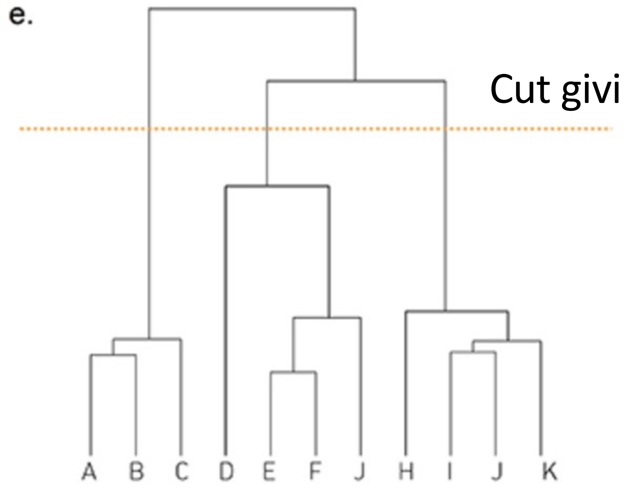
Divisive Procedures: the Girvan-Newman Algorithm



Complex Networks

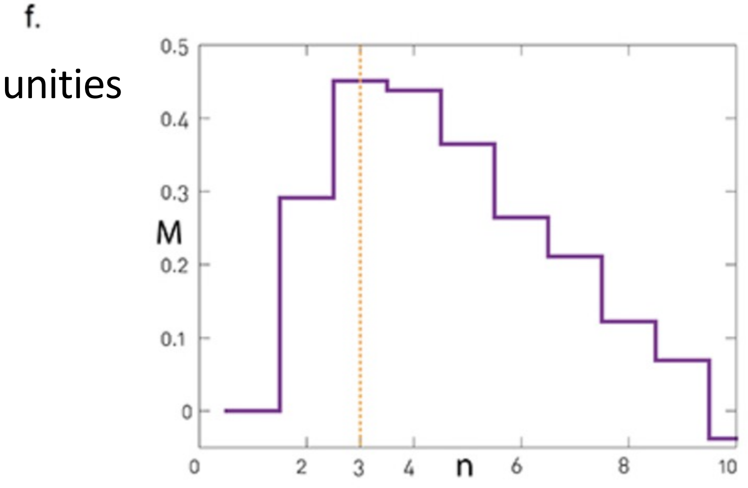
Community Detection

Divisive Procedures: the Girvan-Newman Algorithm



Cut giving 3 communities

Dendrogram



Cut is determined using a Modularity measure M

Divisive Procedures: the Girvan-Newman Algorithm

Computational complexity depends on the centrality metric

For link betweenness: $O(LN)$

Including Modularity: $O(L^2N)$
$O(N^3)$ for sparse graph

Complex Networks

Community Detection

Divisive Procedures: the Girvan-Newman Algorithm

The Girvan-Newman algorithm predicted communities in Zachary's Karate Club that the matched almost perfectly two groups after the break-up.