04-630
Data Structures and Algorithms for Engineers

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Lecture 13

Trees I

- Types of trees
- Binary Tree ADT
- Binary Search Tree
- Height Balanced Trees
  - AVL Trees
  - Red-Black Trees
- Optimal Code Trees
- Huffman's Algorithm

Binary Search Trees

- A Binary Search Tree (BST) is a special type of binary tree
  - it represents information is an ordered format
  - A binary tree is binary search tree if for every node w,

    all keys in the left subtree of w have values less than the key of w

    all keys in the right subtree have values greater than key of w.

Binary Search Trees

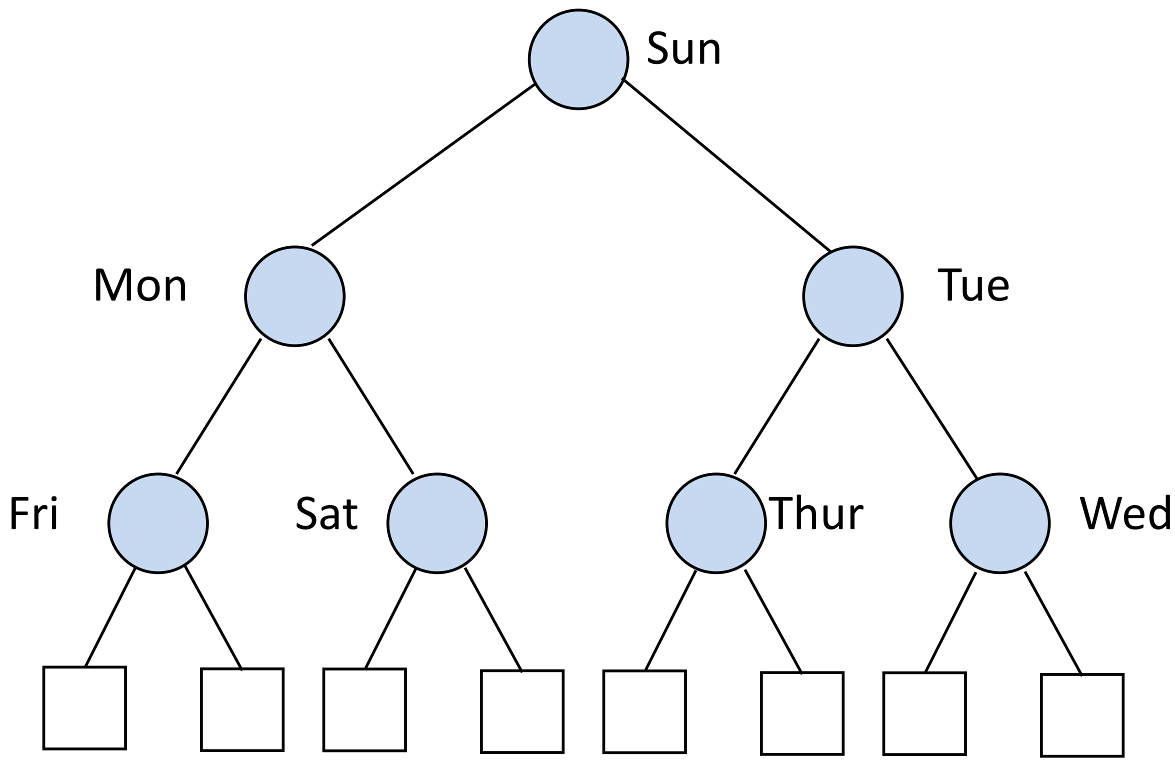
Definition: A binary search tree T is a binary tree; either it is empty or each node in the tree contains an identifier and:

– all keys in the left subtree of T are less (numerically or alphabetically) than the identifier in the root node T;

– all identifiers in the right subtree of T are greater than the identifier in the root node T;

– The left and right subtrees of T are also binary search trees.

Binary Search Trees



Binary Search Trees

- The main point to notice about such a tree is that, if traversed inorder, the keys of the tree (*i.e.* its data elements) will be encountered in a sorted fashion

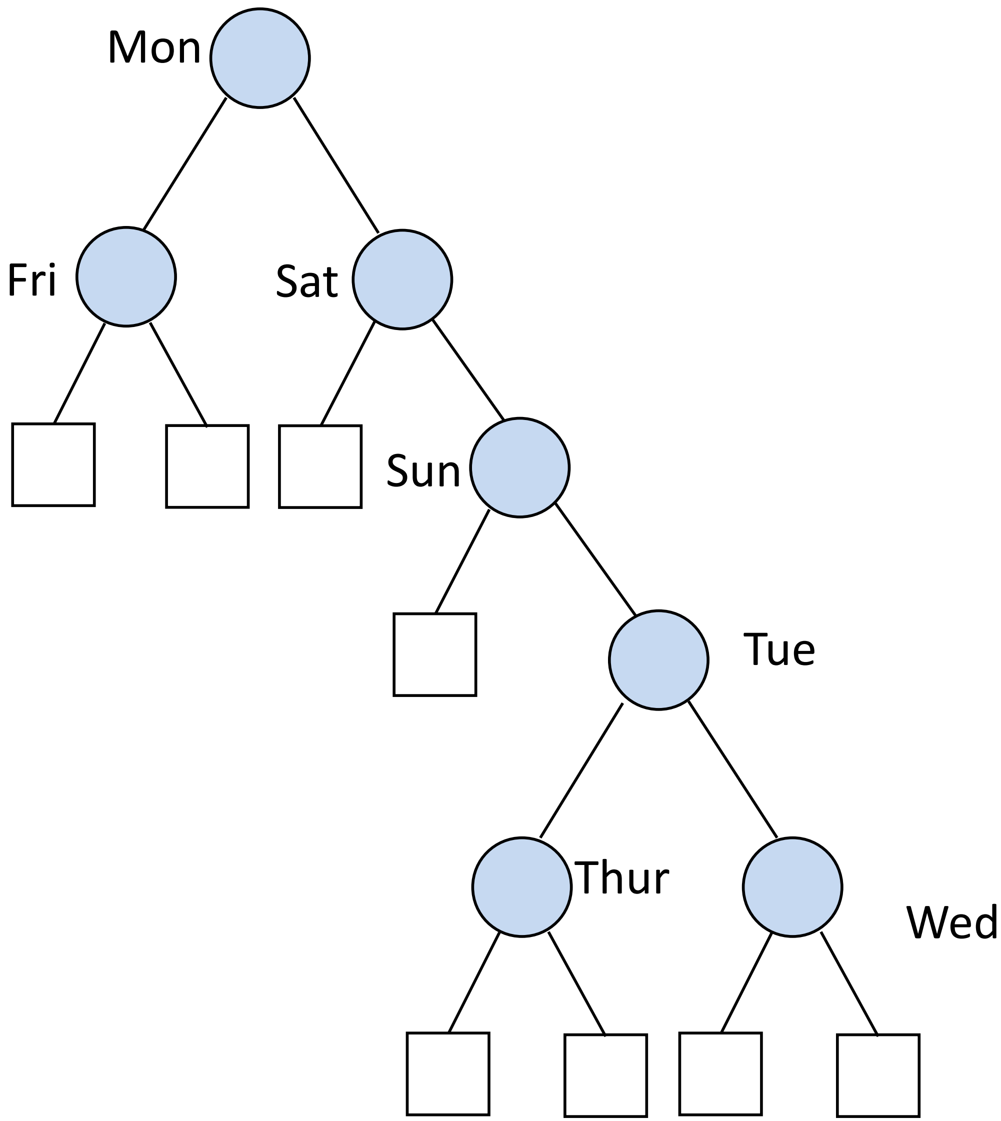
- Furthermore, efficient searching is possible using the binary search technique

  – search time is $O(log_2n)$

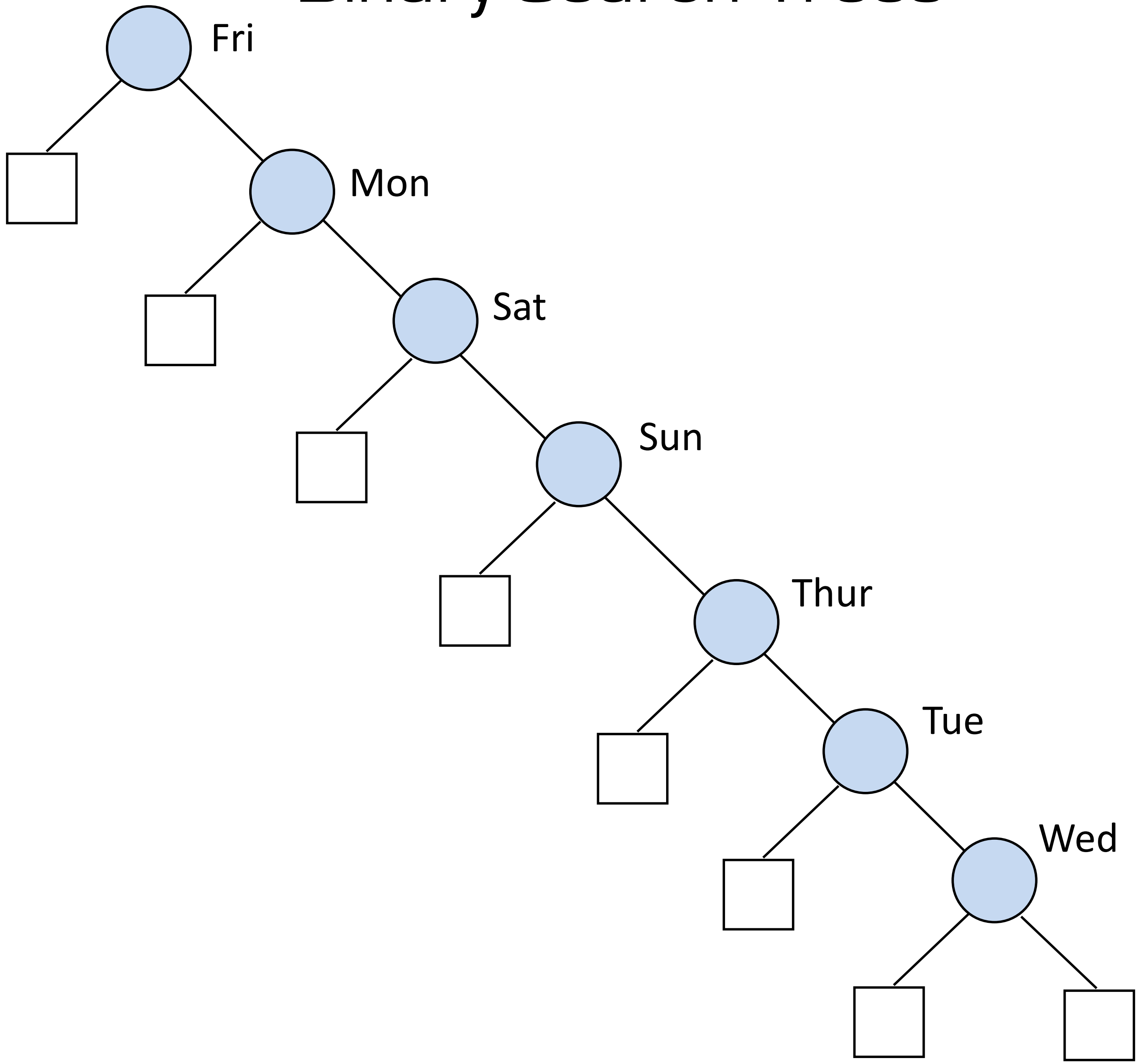
Binary Search Trees

It should be noted that several binary search trees are possible for a given data set, *e.g,* consider the following tree:

Binary Search Trees



Binary Search Trees



Binary Search Trees

Let us consider how such a situation might arise

Construct a binary search tree:

- Assume we are building a binary search tree of words
- Initially, the tree is null, i.e. there are no nodes in the tree
- The first word is inserted as a node in the tree as the root, with no children

Binary Search Trees

On insertion of the second word, we check to see if it is the same as the key in the root, less than it, or greater than it

- If it is the same, no further action is required (duplicates are not allowed)
- If it is less than the key in the current node, move to the left subtree and *compare again*
- If the left subtree does not exist, then the word does not exist and it is inserted as a new node on the left

Binary Search Trees

- If, on the other hand, the word was greater than the key in the current node, move to the right subtree and compare again

- If the right subtree does not exist, then the word does not exist and it is inserted as a new node on the right

– This insertion can most easily be effected in a recursive manner

Binary Search Trees

- The point here is that the structure of the tree depends on the order in which the data is inserted in the list

- If the words are entered in sorted order, then the tree will degenerate to a 1-D list

BST Operations

- $Insert$: E x BST $\rightarrow$ BST :

  The function value $Insert(e,T)$ is the BST T with the element e inserted as a leaf node; if the element already exists, no action is taken

NO WINDOW!!!

BST Operations

- $Delete$: E x BST $\rightarrow$ BST :

  The function value $Delete(e, T)$ is the BST T with the element e deleted; if the element is not in the BST exists, no action is taken.

NO WINDOW!!!

Implementation of $Insert(e, T)$

- If T is empty (i.e. T is NULL)
  - create a new node for e
  - make T point to it
- If T is not empty
  - if e < element at root of T
    - Insert e in left child of T: $Insert(e, T(1))$
  - if e > element at root of T
    - Insert e in right child of T: $Insert(e, T(2))$

Implementation of $Delete(e, T)$

First, we must locate the element e to be deleted in the tree

- if e is at a leaf node
  - we can delete that node and be done
- if e is at an interior node at w
  - we can't simply delete the node at w as that would disconnect its children
- if the node at w has only one child
  - we can replace that node with its child

Implementation of $Delete(e, T)$

– if the node at w has two children

- we replace the node at w with the lowest-valued element among the descendents of its right child

- this is the left-most node of the right tree

- It is useful to have a function $\mathrm{DeleteMin}()$ which
  removes the smallest element from a non-empty tree
  and
  returns the value of the element removed

Implementation of $Delete(e, T)$

- If T is not empty
  - if e < element at root of T

    Delete e from left child of T: *Delete(e, T(1))*
  - if e > element at root of T

    Delete e from right child of T: *Delete(e, T(2))*
  - if e = element at root of T and both children are empty

    Remove T

Implementation of $Delete(e, T)$

– if e = element at root of T and left child is empty

Replace T with $T(2)$

– if e = element at root of T and right child is empty

Replace T with $T(1)$

– if e = element at root of T and neither child is empty

Replace T with left-most node of $T(2)$ ← *"left-most node in right sub-tree!"*

Implementation of $Delete(e, T)$

What is the left-most node in the right sub-tree has two (interior node) children?

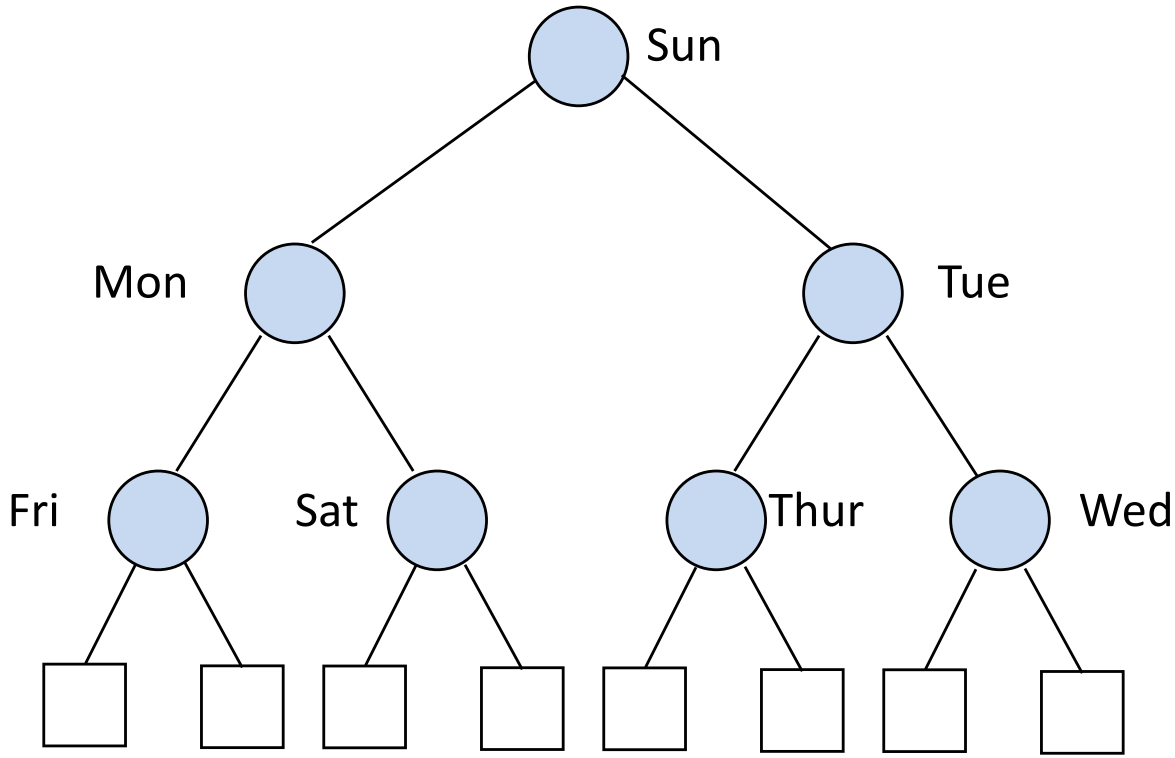
Implementation of $Delete(e, T)$

It can't!

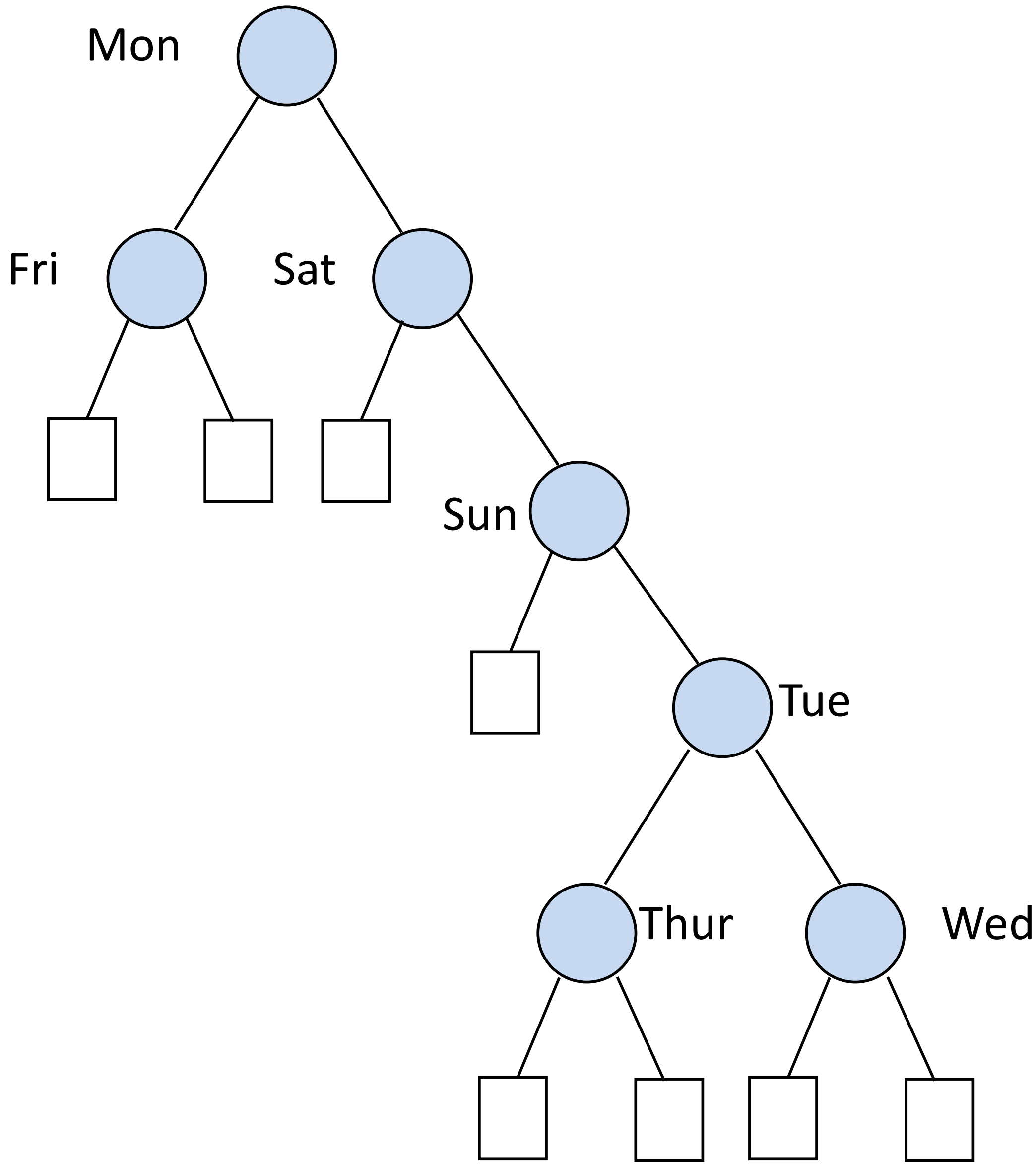
If it did, it wouldn't be the left-most node …

because there would be a node on it's left!

Implementation of $Delete(e, T)$



Implementation of $Delete(e, T)$



Tree Traversals

- To perform a traversal of a data structure, we use a method of visiting every node in some predetermined order

- Traversals can be used
  - to test data structures for equality
  - to display a data structure
  - to construct a data structure of a give size
  - to copy a data structure

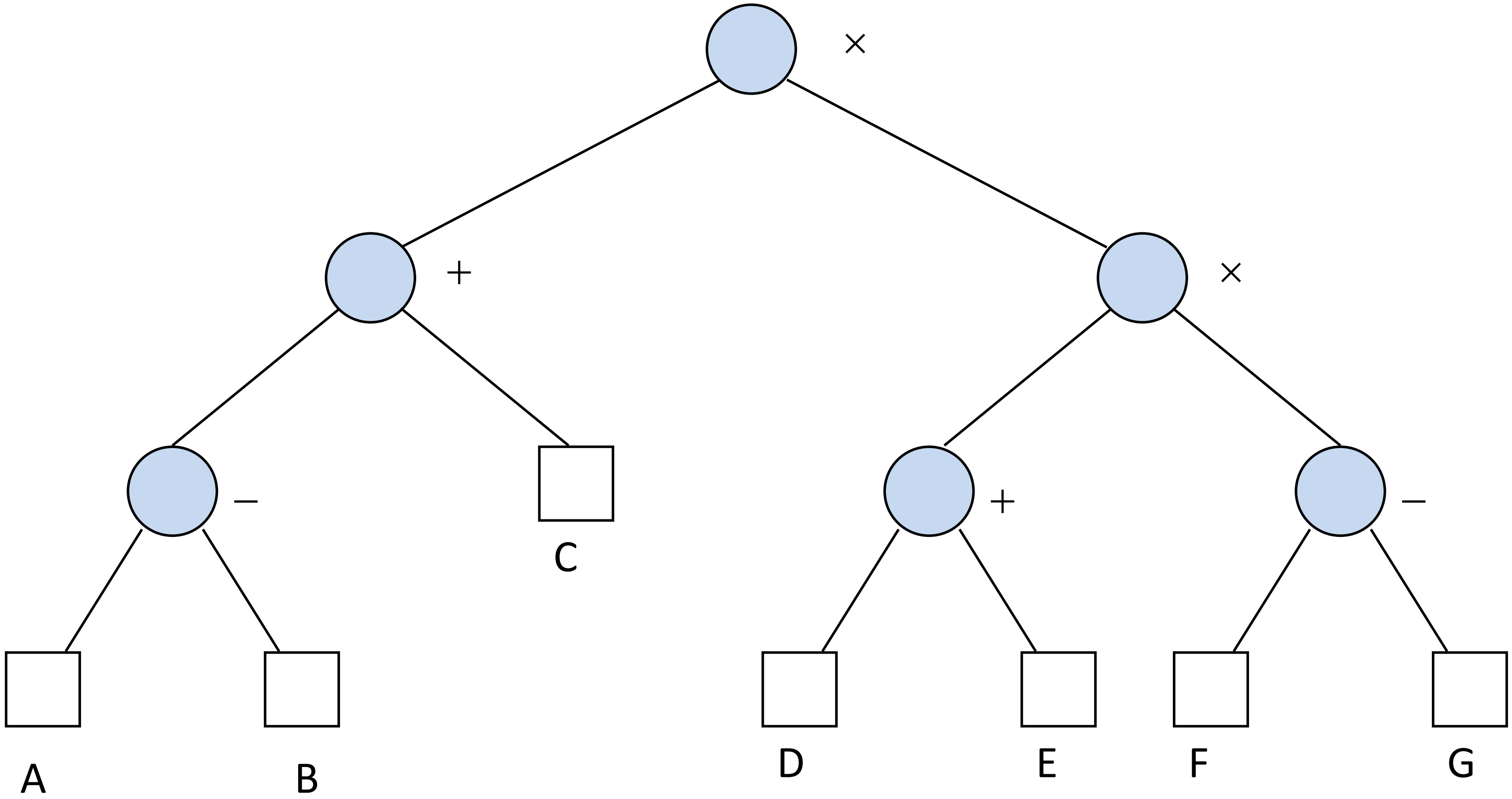
Depth-First Traversals

- There are 3 depth-first traversals

  – Inorder
  – Postorder
  – Preorder

- For example, consider the expression tree:

Example: Expression Tree



Depth-First Traversals

- Inorder traversal

  A – B + C x D + E x F – G

- Postorder traversal

  A B – C + D E + F G – x x

- Preorder traversal

  x + –A B C x + D E – F G

Depth-First Traversals

- The parenthesised Inorder traversal

  ((A – B) + C) x ((D + E) x (F – G))

  This is the infix expression corresponding to the expression tree

- Postorder traversal gives a postfix expression

- Preorder traversal gives a prefix expression

Depth-First Traversals

- Recursive definition of inorder traversal

  Given a binary tree T

  if T is empty

  visit the external node

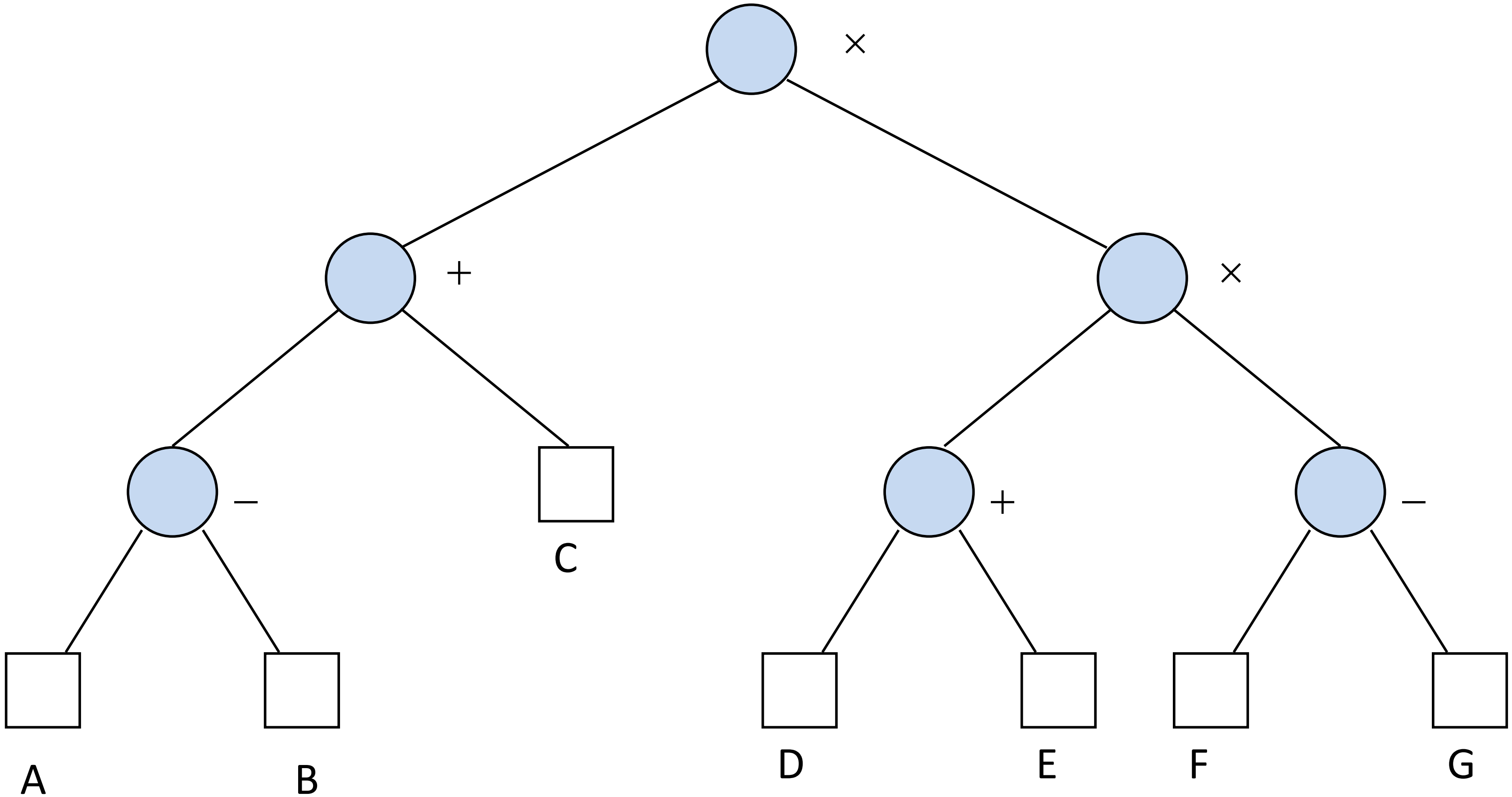
  otherwise

  perform an inorder traversal of $Left(T)$

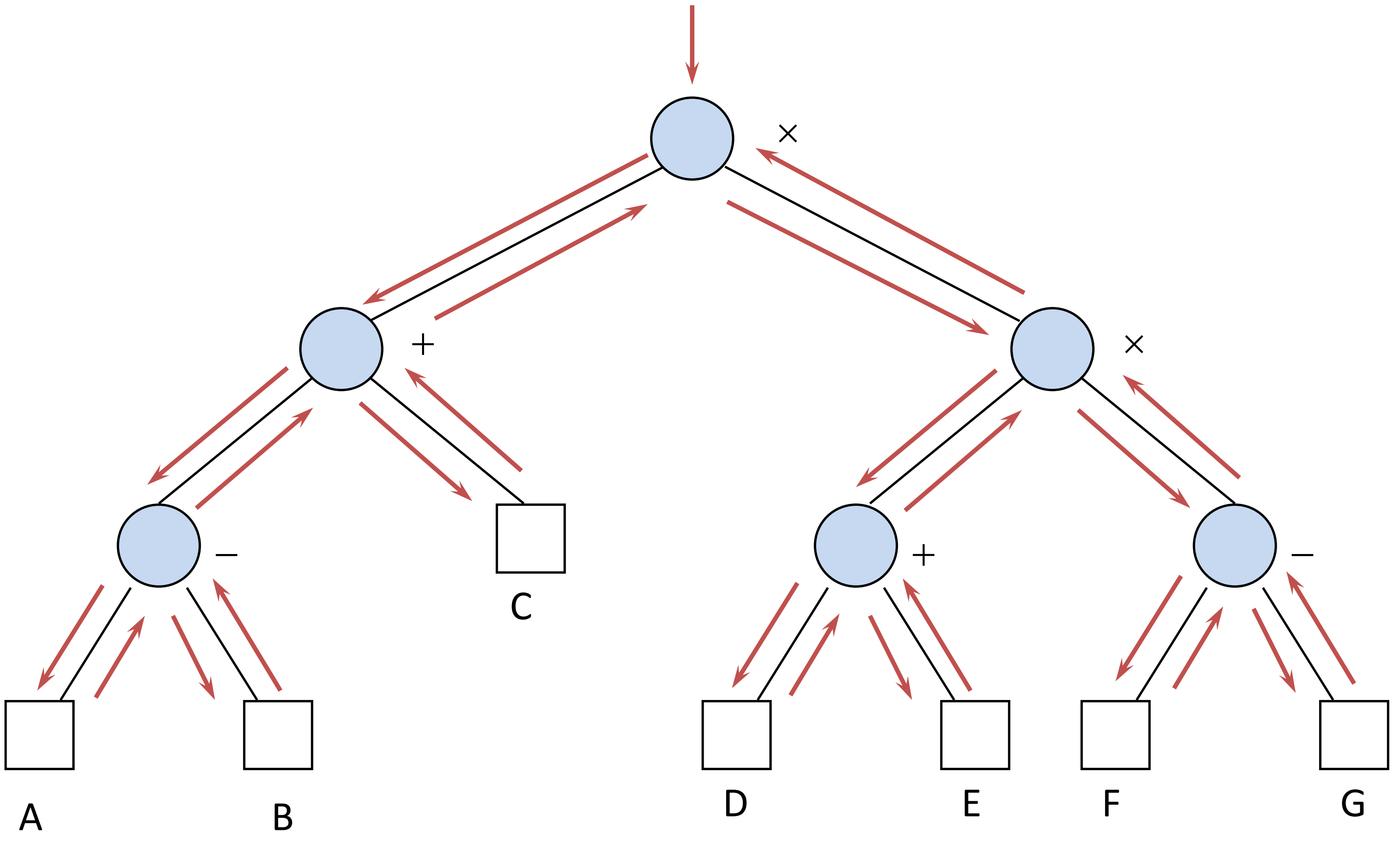
  visit the root of T

  perform an inorder traversal of $Right(T)$

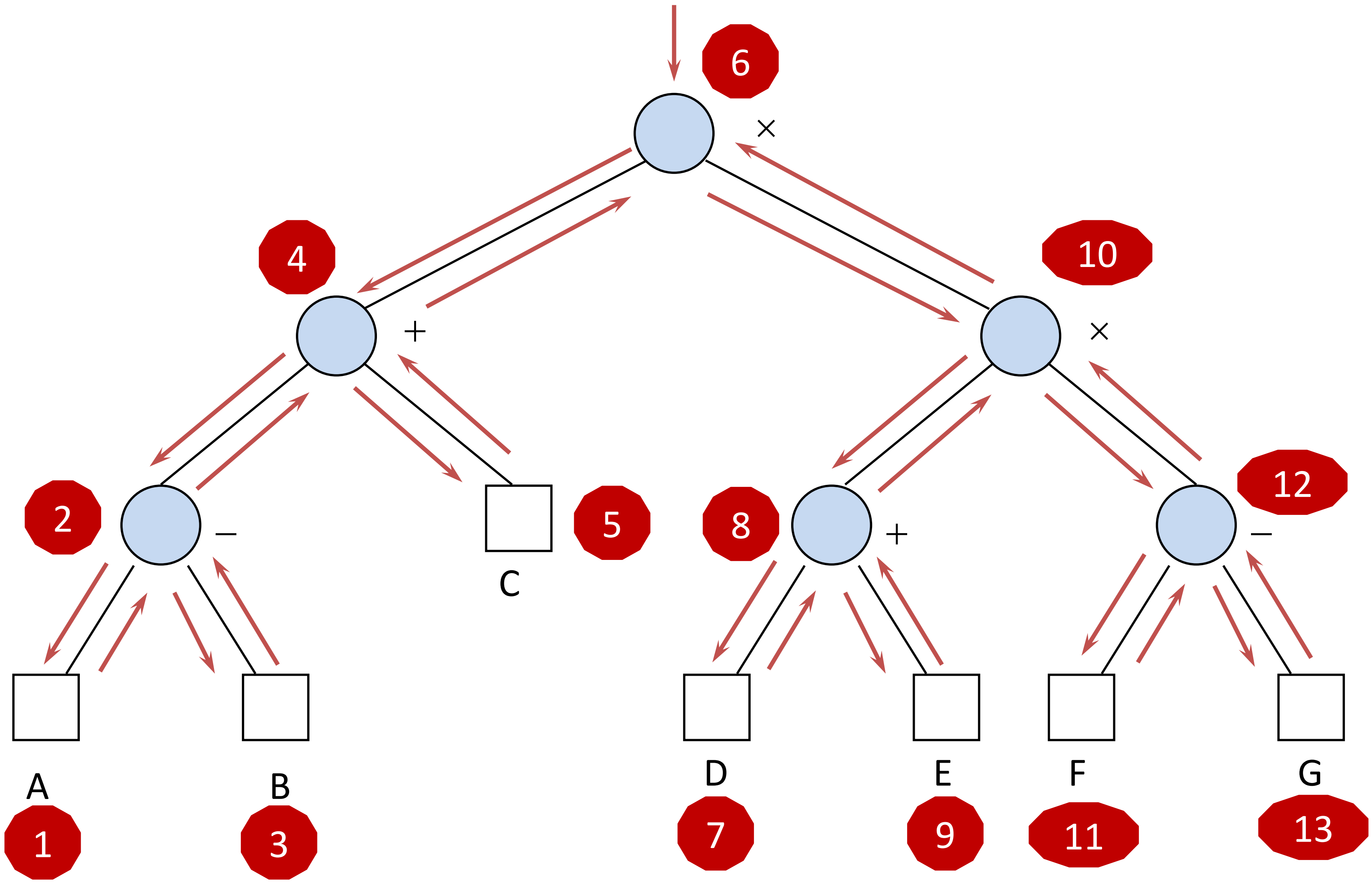
Example: Inorder Traversal



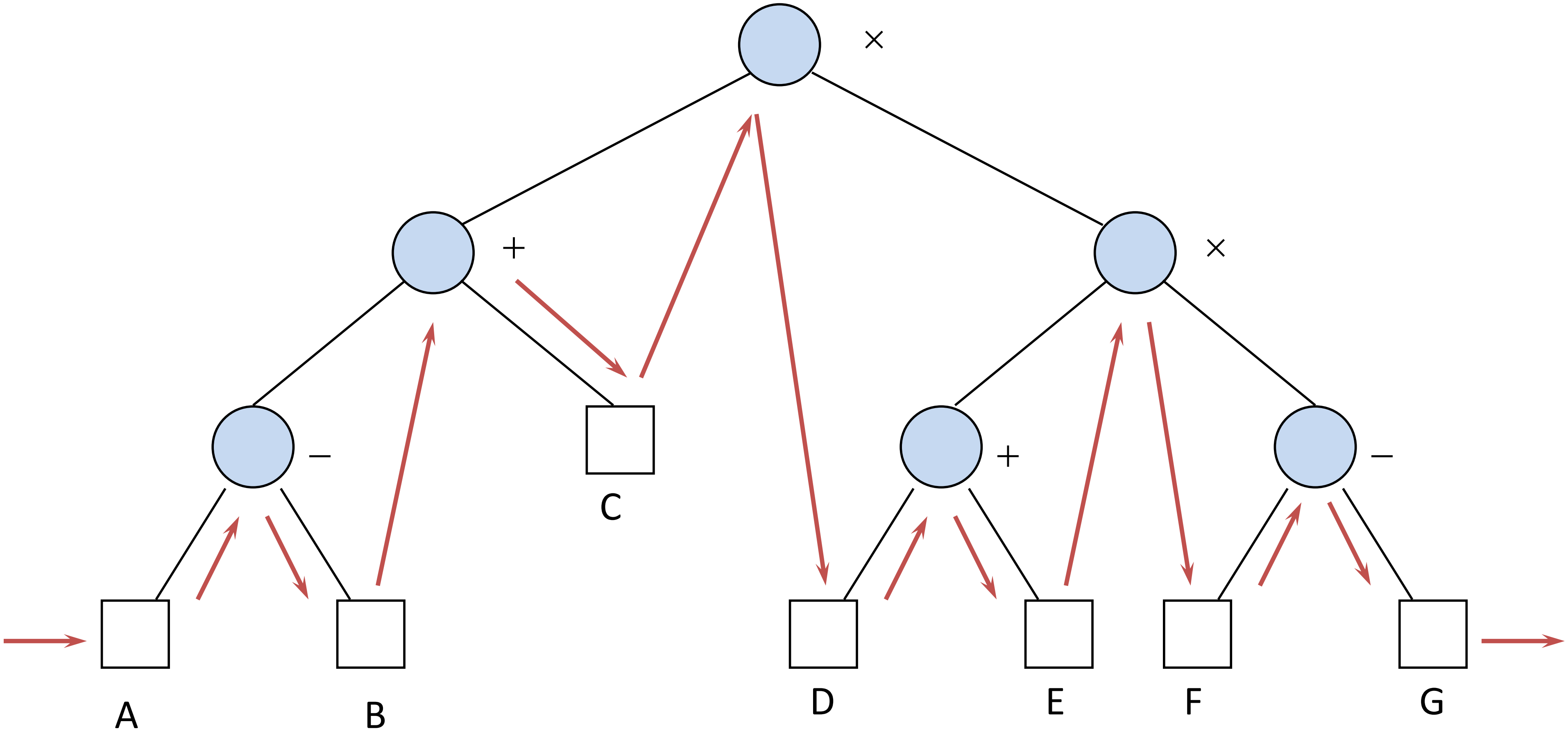
Example: Inorder Traversal



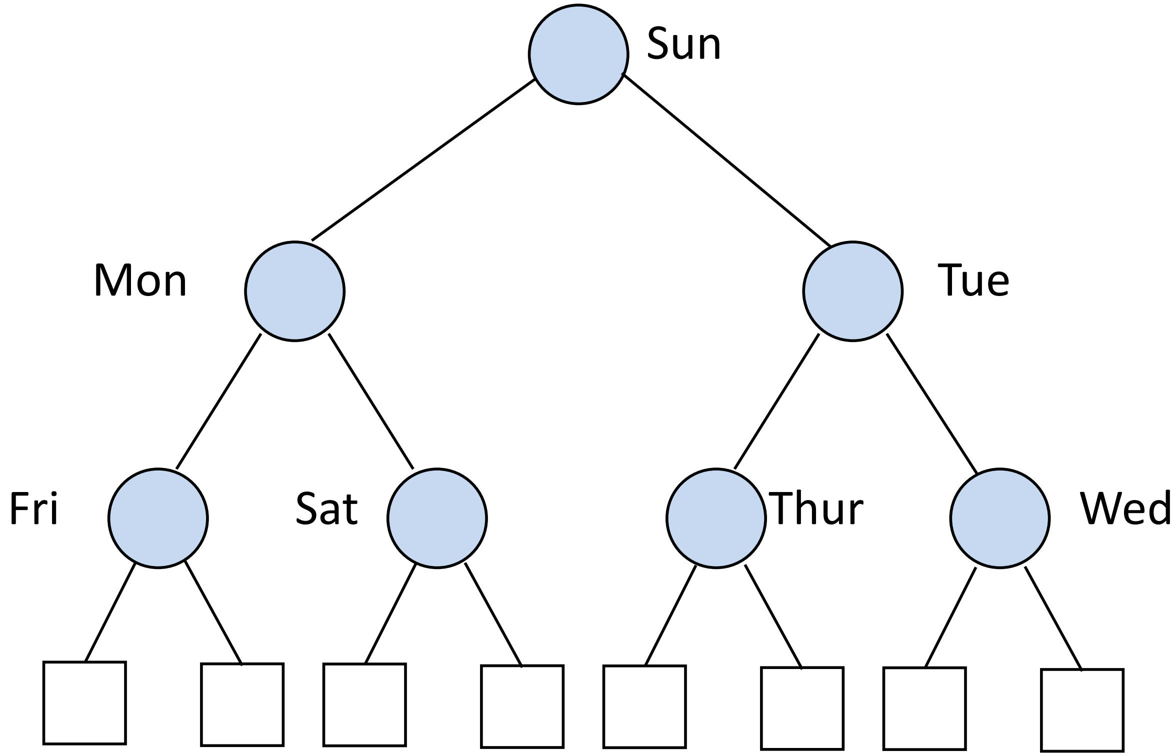
Example: Inorder Traversal



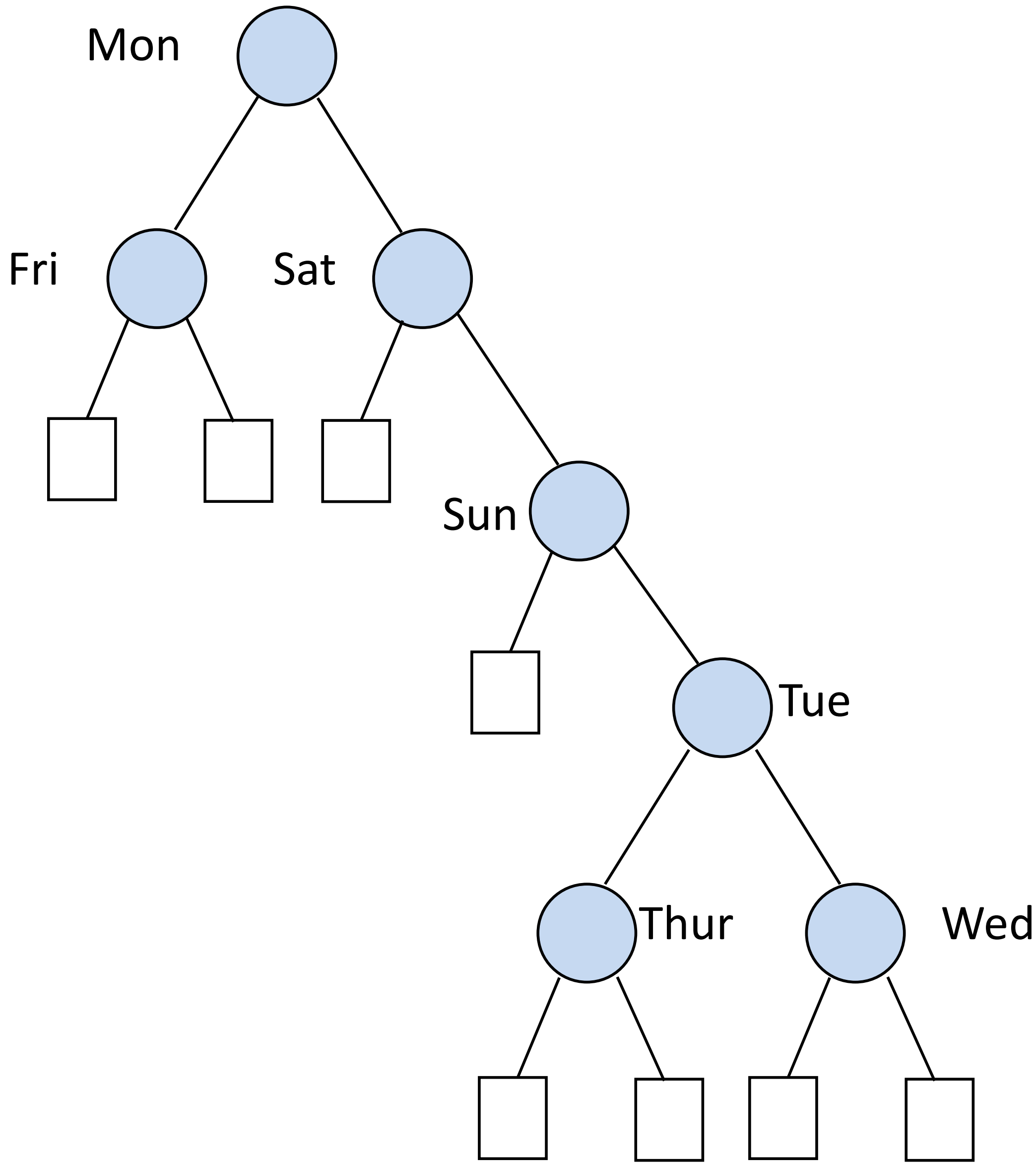
Example: Inorder Traversal



Example: Inorder Traversal



Example: Inorder Traversal



Depth-First Traversals

- Recursive definition of postorder traversal

  Given a binary tree T

  if T is empty

  visit the external node

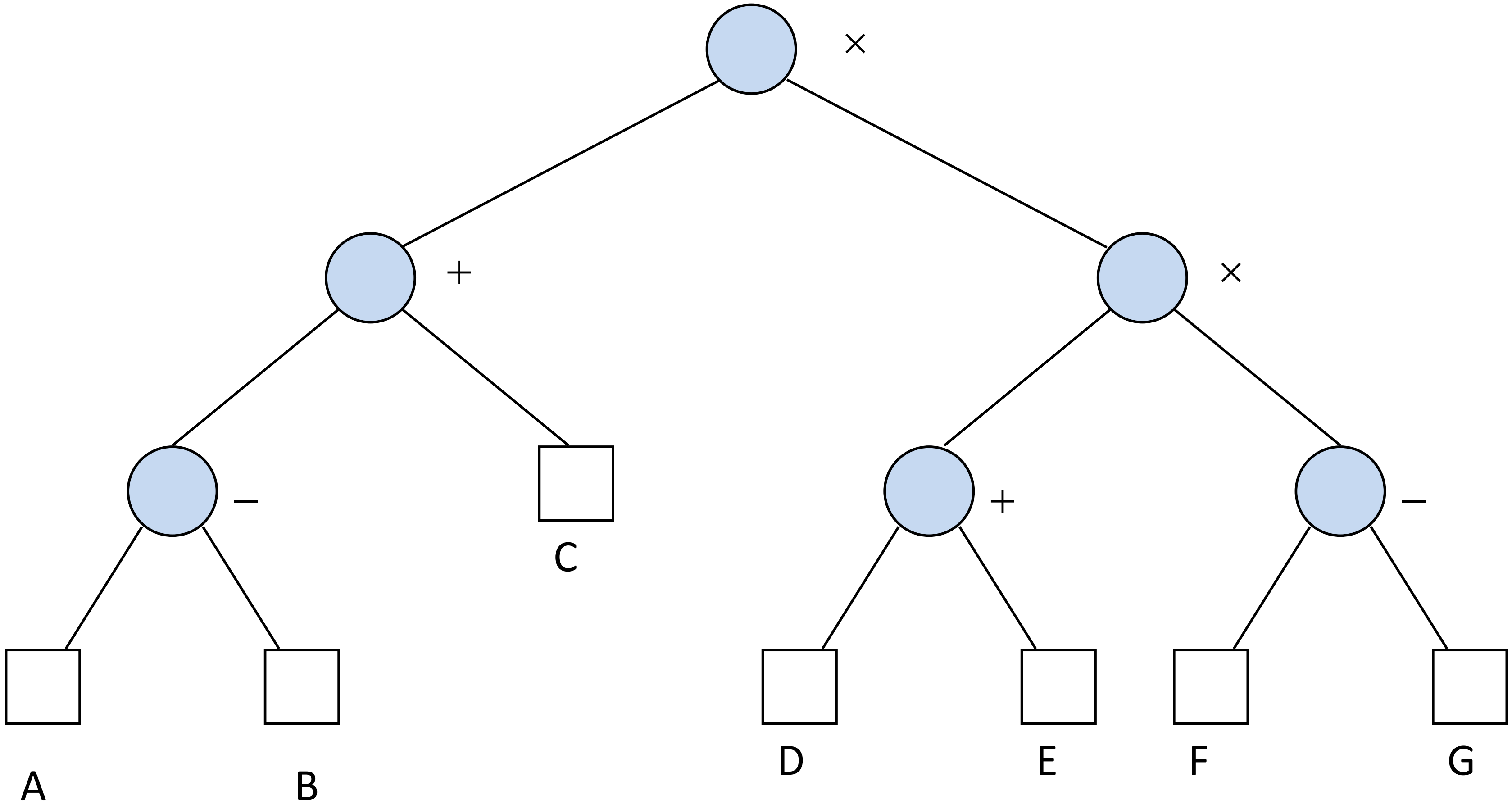
  otherwise

  perform an postorder traversal of $Left(T)$

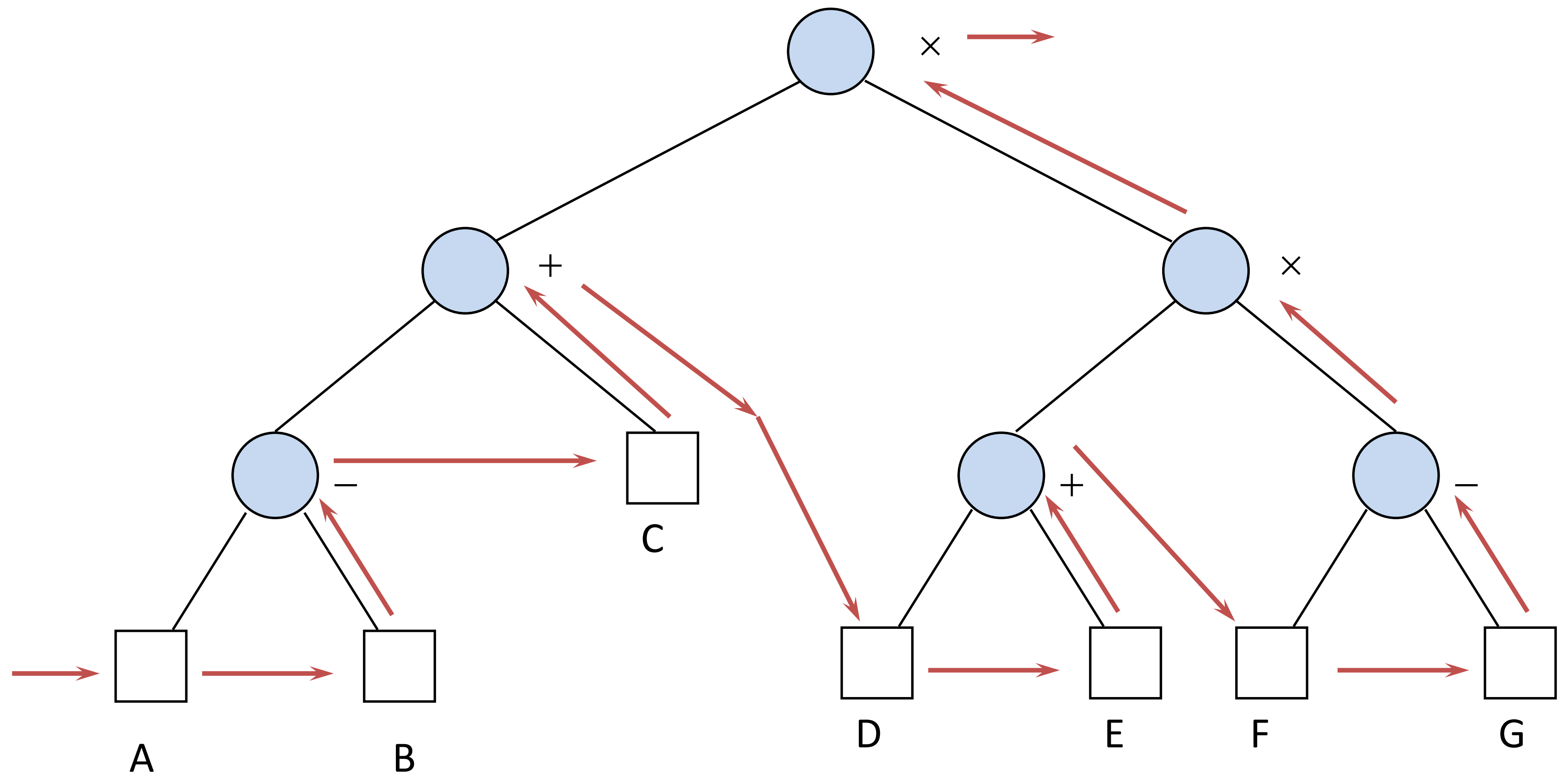
  perform an postorder traversal of $Right(T)$

  visit the root of T

Example: Postorder Traversal



Example: Postorder Traversal



Depth-First Traversals

- Recursive definition of preorder traversal

  Given a binary tree T

  if T is empty

  visit the external node

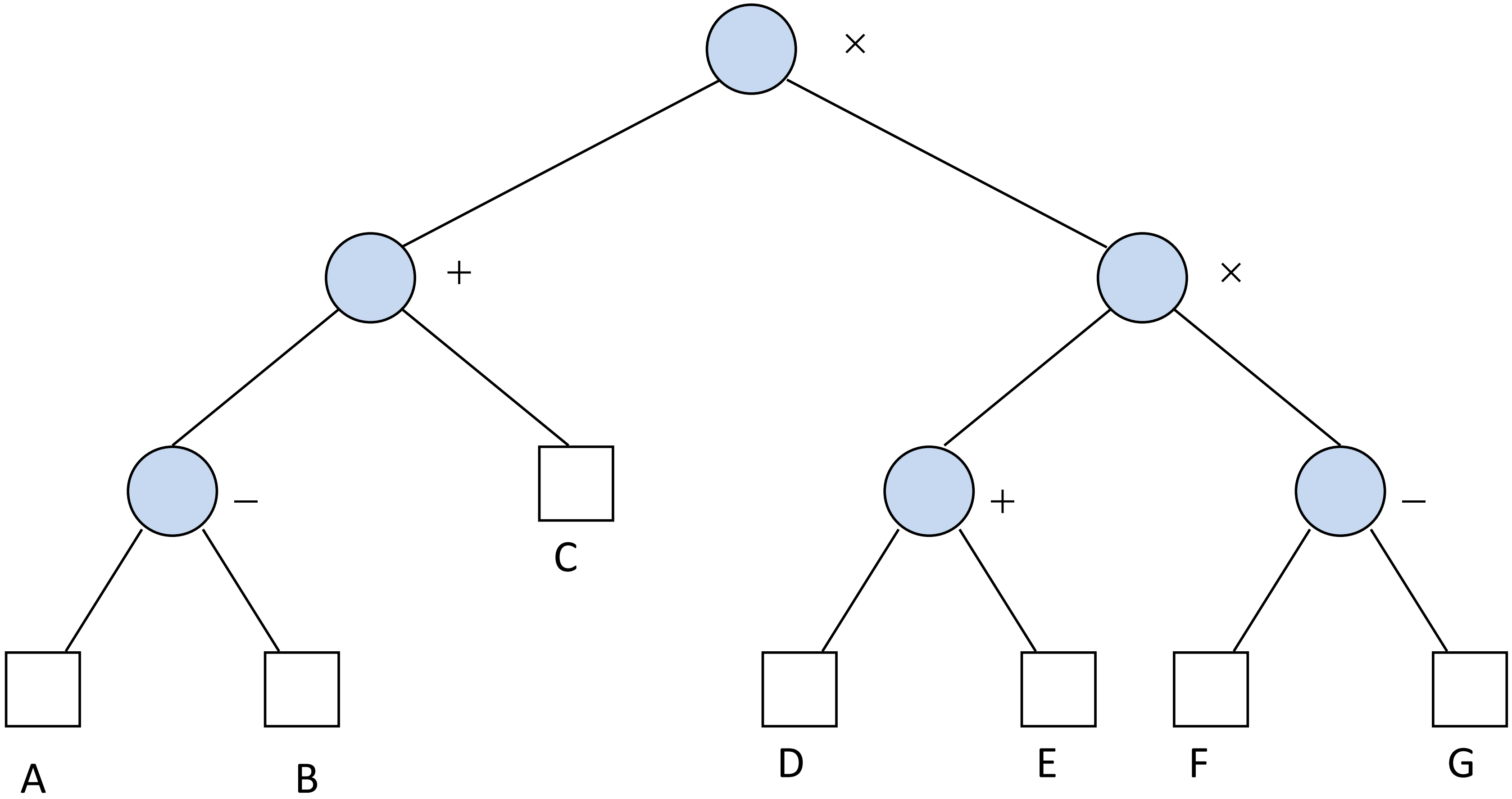
  otherwise

  visit the root of T

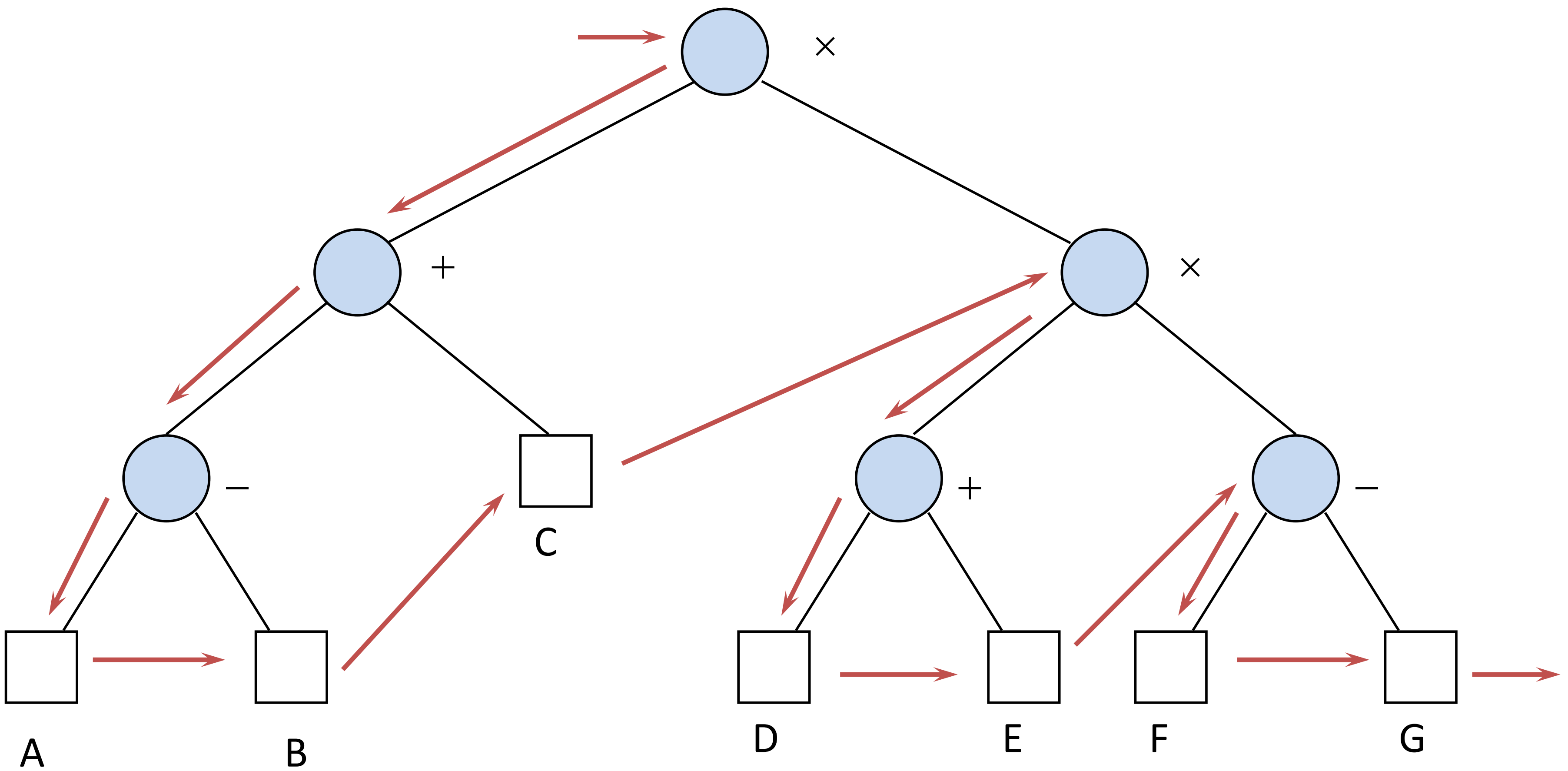
  perform an preorder traversal of $Left(T)$

  perform an preorder traversal of $Right(T)$

Example: Preorder Traversal



Example: Preorder Traversal



BST Implementation

```c
typedef  struct {
            int number;
            char *string;
         } ELEMENT_TYPE;

typedef struct node *NODE_TYPE;

typedef struct node {
            ELEMENT_TYPE element;
            NODE_TYPE left, right;
         } NODE;

typedef NODE_TYPE BINARY_TREE_TYPE;

typedef BINARY_TREE_TYPE WINDOW_TYPE;
```

```c
int main() {

   ELEMENT_TYPE e;
   BINARY_TREE_TYPE tree;

   initialize(&tree);

   print(tree);

   assign_element_values(&e, 3, "...");
   insert(e, &tree);
   print(tree);

   assign_element_values(&e, 1, "+++");
   insert(e, &tree);
   print(tree);

   assign_element_values(&e, 5, "---");
   insert(e, &tree);
   print(tree);

   assign_element_values(&e, 2, ";;;");
   insert(e, &tree);
   print(tree);

   assign_element_values(&e, 4, "***");
   insert(e, &tree);
   print(tree);

   assign_element_values(&e, 6, "000");
   insert(e, &tree);
   print(tree);

   assign_element_values(&e, 3, "...");
   delete_element(e, &tree);
   print(tree);
```

```c
/*** initialize a tree ***/

void initialize(BINARY_TREE_TYPE *tree) {

   static bool first_call = true;

   /* we don't know what value *tree has when the program is launched      */
   /* so we have to be careful not to dereference it                       */
   /* if it's the first call to initialize, there is no tree to be deleted */
   /* and we just set *tree to NULL                                        */

   if (first_call) {
      first_call = false;
      *tree = NULL;
   }
   else {
      if (*tree != NULL) postorder_delete_nodes(*tree);
      *tree = NULL;
   }
}
```

```c
/*** insert an element in a tree ***/

BINARY_TREE_TYPE *insert(ELEMENT_TYPE e,  BINARY_TREE_TYPE *tree ) {

   WINDOW_TYPE temp;

   if (*tree == NULL) {

      /* we are at an external node: create a new node and insert it */

      if ((temp = (NODE_TYPE) malloc(sizeof(NODE))) == NULL)
         error("function insert: unable to allocate memory");
      else {
         temp->element = e;
         temp->left    = NULL;
         temp->right   = NULL;
         *tree = temp;
      }
   }
   else if (e.number < (*tree)->element.number) { /* assume the number field is the key */
      insert(e, &((*tree)->left));
   }
   else if (e.number > (*tree)->element.number) {
      insert(e, &((*tree)->right));
   }

   /* if e.number == (*tree)->element.number, e already is in the tree so do nothing */

   return(tree);
}
```

```c
/*** returns & deletes the smallest node in a tree (i.e. the left-most node) */

ELEMENT_TYPE delete_min(BINARY_TREE_TYPE *tree) {

   ELEMENT_TYPE e;
   BINARY_TREE_TYPE p;

   if ((*tree)->left == NULL) {

      /* tree points to the smallest element */

      e = (*tree)->element;

      /* replace the node pointed to by tree by its right child */

      p = *tree;
      *tree = (*tree)->right;
      free(p);

      return(e);
   }
   else {

      /* the node pointed to by tree has a left child */

      return(delete_min(&((*tree)->left)));
   }

}
```

```c
/*** delete an element in a tree ***/

BINARY_TREE_TYPE *delete_element(ELEMENT_TYPE e, BINARY_TREE_TYPE *tree) {

   BINARY_TREE_TYPE p;

   if (*tree != NULL) {

      if (e.number < (*tree)->element.number)  /* assume element.number is the */
         delete_element(e, &((*tree)->left));  /* key                          */

      else if (e.number > (*tree)->element.number)
         delete_element(e, &((*tree)->right));

      else if (((*tree)->left == NULL) && ((*tree)->right == NULL)) {

         /* leaf node containing e - delete it */

         p = *tree;
         free(p);
         *tree = NULL;
      }
```

```
    else if ((*tree)->left == NULL) {

       /* internal node containing e and it has only a right child */
       /* delete it and make treepoint to the right child          */

       p = *tree;
       *tree = (*tree)->right;
       free(p);
    }
    else if ((*tree)->right == NULL) {

       /* internal node containing e and it has only a left child */
       /* delete it and make treepoint to the left child          */

       p = *tree;
       *tree = (*tree)->left;
       free(p);
    }
    else {

       /* internal node containing e and it has both left and right child */
       /* replace it with leftmost node of right sub-tree                 */
       (*tree)->element = delete_min(&((*tree)->right));
    }
  }
  return(tree);
}
```

```c
/*** inorder traversal of a tree, printing node elements **/

int inorder(BINARY_TREE_TYPE tree, int n) {

   int i;

   if (tree != NULL) {
      inorder(tree->left, n+1);

      for (i=0; i<n; i++) printf("        ");
      printf("%d %s\n", tree->element.number, tree->element.string);

      inorder(tree->right, n+1);
   }
   return(0);
}
```

```c
/*** inorder traversal of a tree, deleting node elements **/

int postorder_delete_nodes(BINARY_TREE_TYPE tree) {

   if (tree != NULL) {
      postorder_delete_nodes(tree->left);
      postorder_delete_nodes(tree->right);
      free(tree);
   }
   return(0);
}
```

```c
/*** print all elements in a tree by traversing inorder ***/

int print(BINARY_TREE_TYPE tree) {

   printf("Contents of tree by inorder traversal: \n");

   inorder(tree,0);

   printf("--- \n");

   return(0);
}
```

```
/*** error handler:
     print message passed as argument and take appropriate action ***/

int error(char *s) {

   printf("Error: %s\n",s);

   exit(0);
}
```

```c
/*** assign values to an element ***/

int assign_element_values(ELEMENT_TYPE *e, int number, char s[]) {

   e->string = (char *) malloc(sizeof(char) * (strlen(s)+1));
   strcpy(e->string, s);
   e->number = number;
   return(0);
}
```

```c
int main() {

   ELEMENT_TYPE e;
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   initialize(&tree);

   print(tree);

   assign_element_values(&e, 3, "...");
   insert(e, &tree);
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   insert(e, &tree);
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   assign_element_values(&e, 6, "000");
   insert(e, &tree);
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   assign_element_values(&e, 3, "...");
   delete_element(e, &tree);
   print(tree);
```
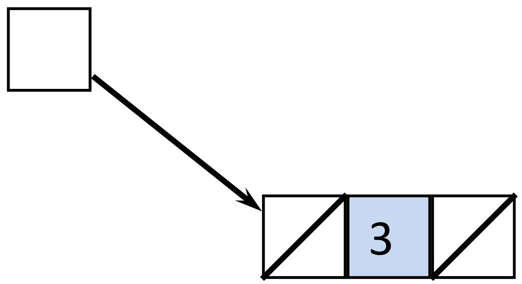

BINARY_TREE Implementation



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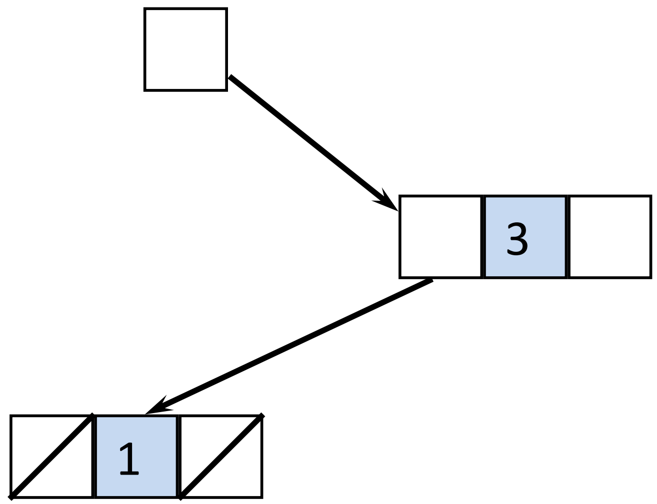

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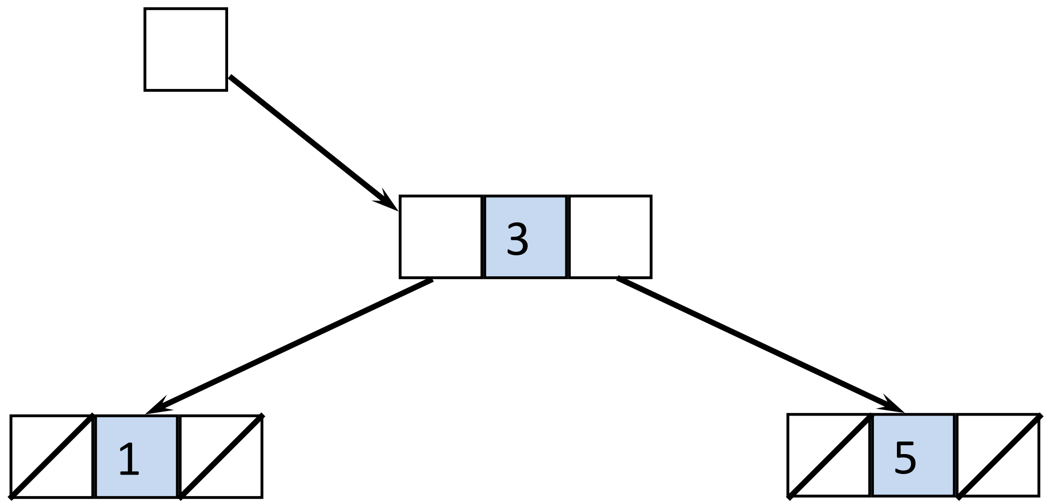

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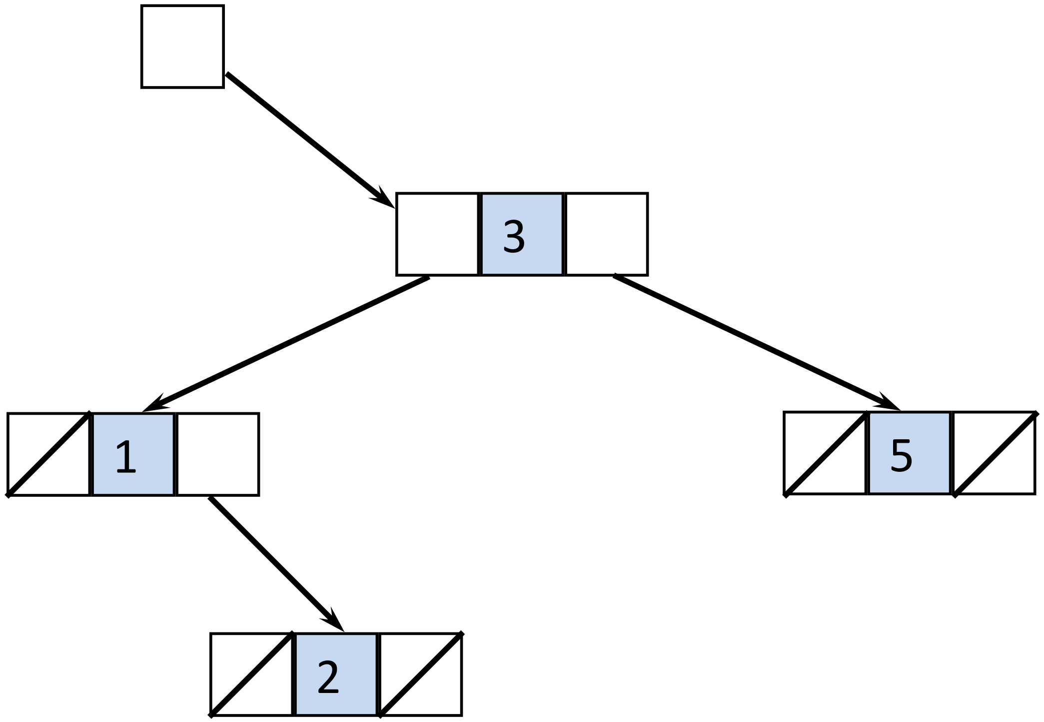
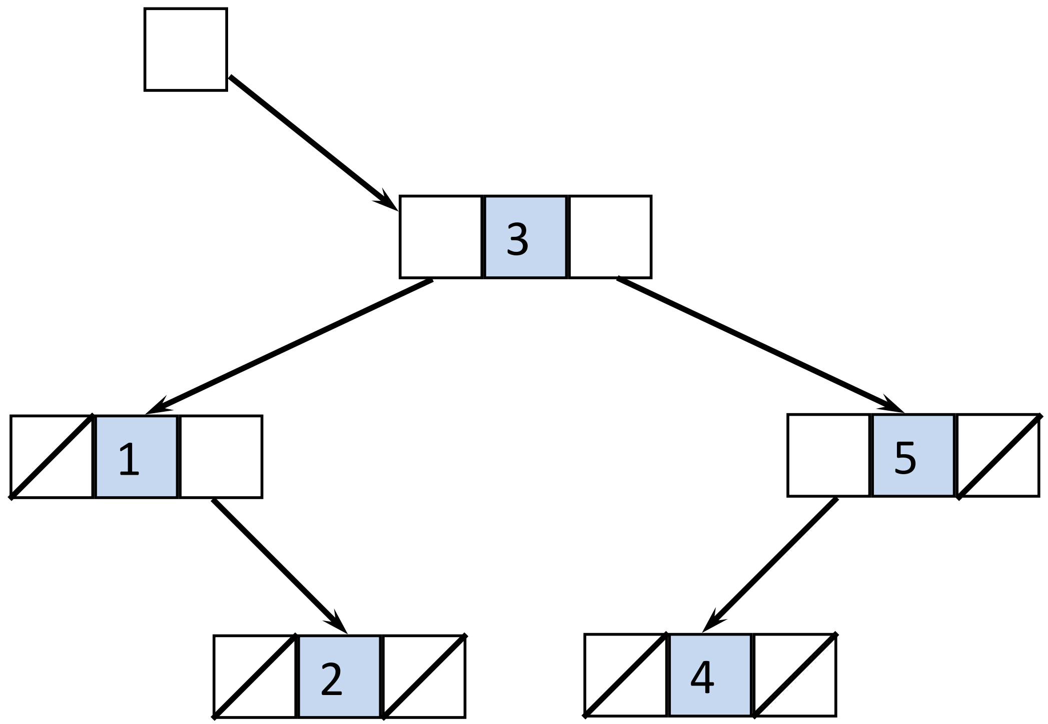
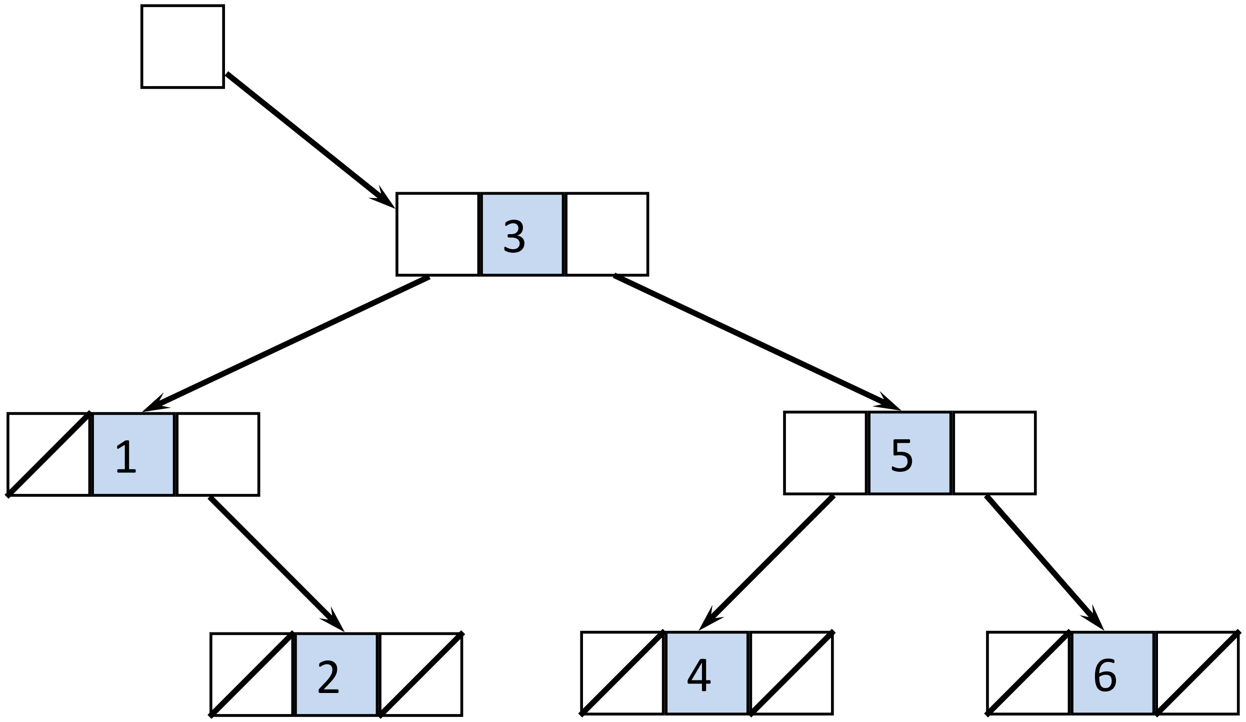


BINARY_TREE Implementation



BINARY_TREE Implementation



BINARY_TREE Implementation